# Mohammed Abouzaid - Symplectic topology and non-archimedean geometry Part 1

Thursday, July 7, 2016 11:02 AM

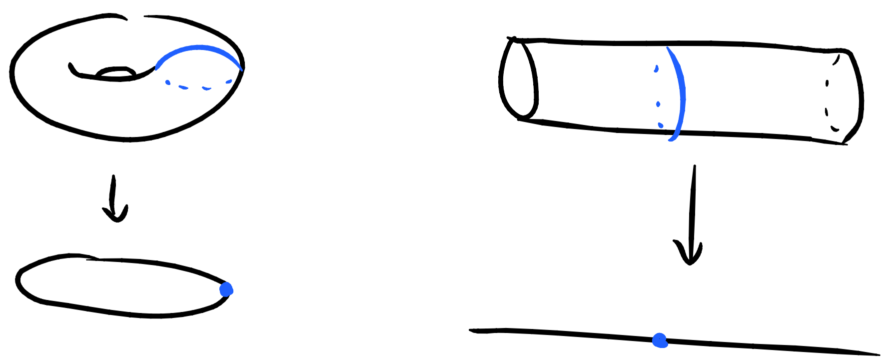
**Motivation:**

$$\begin{tikzcd} X \arrow[d, "\pi"] \\ Q \end{tikzcd}$$

$X$ symplectic

Lagrangian torus fibration

( e.g.



$$\begin{tikzcd} T^{*}T^n \arrow[d] \\ \mathbb{R}^n \end{tikzcd}$$

)

**SYZ:**

$$\begin{tikzcd} X \arrow[rr, leftrightarrow, "\text{mirror}"] \arrow[dr] & & Y \arrow[dl] \\ & Q & \end{tikzcd}$$

$Y$ = $\mathbb{C}$-mfld whose pts are the rk 1 unitary local systems on fibres of $X \to Q$

$X$ comes in a family (rescaling $\omega$)

$\hookrightarrow$ Family $Y_t$

" variety over $\mathbb{C}[t]$ "

Kontsevich, Fukaya:

Floer theoretically, this requires convergence of operations in the Fukaya category

e.g: In $CF(L_0, L_1)$ for $L_0 \subset X$,

the diff'l should look like

$$\sum_{u \text{ rigid}} e^{-\mathrm{Area}(u)/\hbar}$$

Gromov compactness

$$\Longrightarrow \sum T^{\mathrm{Area}(u)}$$

converges T-adically

We should think of $Y$ as an analytic space over the Novikov field:

(

$\left( \Bbbk = \text{field}, \quad \Lambda := \left\{ \sum_{i=0}^{\infty} a_i T^{\lambda_i} \;\middle|\; \begin{array}{c} a_i \in \Bbbk \\ \lambda_i \to +\infty \end{array} \right\} \right)$

**Goal:** Build tools so that we can assign to each coherent sheaf on $Y$ an object of the "Fukaya category" of $X$

**Toy Case:** $X = T^* \mathbb{T}^n$

mirror: $Y = (\Lambda^*)^n \quad (\Lambda^* = \Lambda \setminus \{0\})$

(i.e. think of $(\mathbb{C}^*)^n$)

- Fukaya category is known to be generated by a fibre, and the self-Floer $H^*$ is isomorphic to

$$\Lambda[\pi_1 \mathbb{T}^n] = \left\{ \begin{array}{c} \text{Laurent polynomials} \\ \text{in } n \text{ variables} \end{array} \right\}$$

$\Lambda$

- A complex of coherent sheaves on $Y$ is a finitely generated complex of modules over this ring.

Build a corresponding complex in Fukaya category (e.g. twisted complex)

e.g: $T^*S^1$

$$\Lambda[z,z^{-1}] \xrightarrow{*(z-1)} \Lambda[z,z^{-1}] \xrightarrow{\text{cokernel}} \Lambda_{z=1}$$

$$f \longmapsto (z-1)f$$

$$0\text{-section} \longleftrightarrow \{z=1\}$$

So expect 0-section is "built from" $T^*_{pt}S^1$

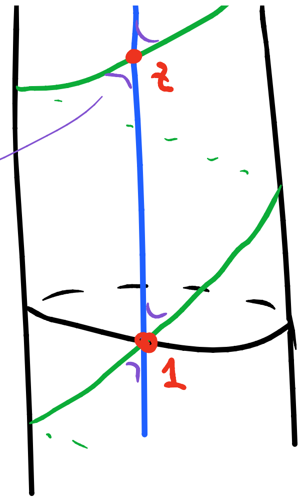
$$T^*_{pt}S^1 \xrightarrow{z-1} T^*_{pt}S^1$$

$$HF^*(T^*_{pt}S^1, T^*_{pt}S^1) \cong \Lambda[z,z^{-1}]$$

$$z \in$$

Geometrically

Lag'n surgery
$\rightsquigarrow$ get circle parallel to 0-section




Another way of assigning to a complex of coherent sheaves on $Y$ an "object of $\mathcal{F}uk(X)$" is to use only the 0-section.

To do this, we allow as objects of $\mathcal{F}uk(X)$ arbitrary local systems on closed Lag'n.

In particular, $X = T^*\mathbb{T}^n$.

We can equip the zero section $\mathbb{T}^n$ with the local system corresponding to $\Lambda[\pi_1 \mathbb{T}^n]$ — call it $\mathcal{U}$

$d$ $1$

as a module over itself.

$N$ = any top. space with $*$

Then $\Lambda\left[\pi_0\left(\substack{\text{space of paths} \\ * \text{ to } X}\right)\right]$ forms a local system

with corresponding repn given by $\Lambda[\pi_1 X]$

The object $(\mathbb{T}^n, \underline{\underline{\mathcal{U}}})$ of the Fukaya category has endo$^m$ algebra

$$HF^*((\mathbb{T}^n, \mathcal{U}), (\mathbb{T}^n, \mathcal{U})) \cong H^*(\mathbb{T}^n, Hom(\mathcal{U}, \mathcal{U}))$$

$$\cong \Lambda[\pi_1 \mathbb{T}^n]$$

## Advantage of Second Approach:

$X$ = any symplectic mfld

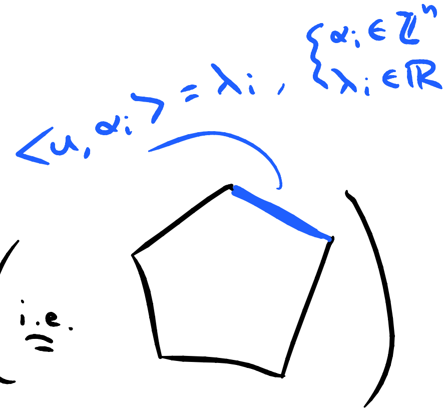
$\cup$

Rmk: $\cancel{\mathbb{T}^n}$ = Lagn ~~to~~ $s$ $\rightsquigarrow$ $(\cancel{\mathbb{T}^n}^{\,L}, \mathcal{U})$

$\mathbb{T}$ - Lagn ~~torus~~ $L$ $\rightsquigarrow$ $(\mathbb{T}, U)$ can be thought of as an "object of $\mathcal{F}(X)$"

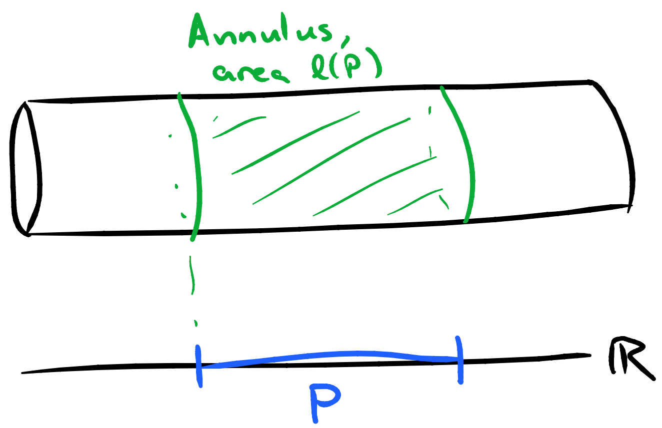
## Local computation:

Consider integral affine polytope $P \subset \mathbb{R}^n$ (i.e.




)

$$\begin{array}{ccc} X_P & \subset & X = T^*\mathbb{T}^n \\ \downarrow & & \downarrow \\ P & \subset & \mathbb{R}^n \end{array}$$




Mirror:

$$\begin{array}{ccc} Y_P & \subset & (\Lambda^*)^n \\ \downarrow & & \downarrow \\ P & \subset & \mathbb{R}^n \end{array}$$

"an analytic space"

Over $\mathbb{C}$:

$$\begin{tikzcd} Y_p^{\mathbb{C}} \arrow[d] & \subset & (\mathbb{C}^*)^n \arrow[d] \\ P & \subset & \mathbb{R}^n \end{tikzcd} \qquad z \mapsto \log|z|$$

convex Stein domain

What is analogue?

$$\Lambda^* \xrightarrow{\text{val}} \mathbb{R}$$

$$\sum a_i T^{\lambda_i} \longmapsto \lambda_0$$

$\lambda_0 < \lambda_1 < \cdots$

$a_0 \neq 0$

This defines $Y_p$ as a set.

The ring of functions on $Y_p$ is the space of Laurent series which converge ($T$-adically) at all points $y \in Y_p$

$$\sum_{\alpha \in \mathbb{Z}^n} c_\alpha \cdot z^\alpha, \quad c_\alpha \in \Lambda$$

i.e. for $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n$,

$z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$

Let $\Gamma_P$ denote the ring of functions.

E.g. $P = \{0\}$

$y \in Y_P$ iff $\quad y_i = \underset{\mathbb{K}^*}{\overset{\cap}{a^i}} + \left(\begin{matrix}\text{higher order} \\ \text{in } T\end{matrix}\right)$

So convergence of $\sum c_\alpha z^\alpha$ at $z = y$ is the same as convergence of $\sum c_\alpha$,

i.e. $\boxed{\lim_{|\alpha| \to \infty} \operatorname{val} c_\alpha = +\infty}$

Floer-theoretic interpretation:

① [Seidel] There is a "quantitative" version of wrapped Floer cohomology for Liouville domains...

Instead of taking $\bigoplus_{\text{Intersection points}}$, take a completion with respect to the action filtration.

This version of HF depends on domain.

([S. Venkatesh]: in non-compact setting, the answer is zero or non-zero depending on the "size" of the domain.)

② Take the completion $\mathcal{U}_p$ of this local system $\mathcal{U}$ which corresponds to the completion $\Gamma_p$ of Laurent polynomials.

i.e. $\Gamma_p$ is naturally a module of

$$\Gamma = \Lambda[z_i^{\pm 1}] \cong \Lambda[\pi_1 T^n]$$

and hence gives rise to a local system.

This completion can be expressed geometrically in terms of the minimal length of repns of any given htpy class.

__Problem__: $\Gamma_p \longrightarrow H^*(\mathbb{T}^n, \mathrm{Hom}(\mathcal{U}_p, \mathcal{U}_p))$

is NOT an isomorphism.

On the mirror side, this reflects the fact that sheaf theory on rigid analytic spaces requires sheaf theory.

What this means

In algebraic geometry, we can compute Hom between quasi-coherent sheaves by using Čech covers by affines.

Moreover if $W \overset{\text{affine}}{\subset} Y$ then $i_*\mathcal{O}_W$ gives a quasi-coherent sheaf.

The same does not work in "rigid analytic geometry" in the sense that if $Y_0 \subset Y_1$ "affinoid" then...

$$\mathrm{Hom}_{\Gamma_{Y_1}}(\Gamma_{Y_0}, \Gamma_{Y_0}) \neq \mathrm{Hom}_{\substack{\text{sheaves}\\ \text{on } Y_1}}(i_*\mathcal{O}_{Y_0}, i_*\mathcal{O}_{Y_0})$$

For coherent sheaves, we can use Čech complexes

Consequence of Tate acyclicity.

One solution: Take into account the topology of local systems.

$\Gamma_p$ is a Banach space$_{/\Lambda}$ with norm

$$\|f\| := \max_{y \in Y_p} e^{-\mathrm{val}(f(y))}$$

So we can take

$$\Gamma_p \cong H^*(\mathbb{T}^n, \mathrm{Hom}_\Lambda^{\mathrm{cts}}(\mathcal{U}_p, \mathcal{U}_p))$$

(

(Explain tomorrow)