Abouzaid - Part 2

$\Gamma := \Lambda[z, z^{-1}]$ (or more generally in higher dims)

$= \text{group ring of } \pi_1(\mathbb{T}^n)$

$\rightsquigarrow$ Local system on $\mathbb{T}^n$

$\mathcal{U}$ (universal)

$$\left(\begin{array}{c} \text{"}\mathcal{U}_x = H_*(\Omega_{x*}S^1)\text{"} \\ x = * \Rightarrow \mathcal{U} \text{ canonically } \Gamma \\ \text{Otherwise, htpy class} \\ \text{of paths } * \to x \text{ gives} \\ \Gamma \cong \mathcal{U}_x \end{array} \right)$$

$$\Gamma \cong H^*(\mathbb{T}^n, \mathrm{Hom}_{\mathbb{k}}(\mathcal{U}, \mathcal{U}))$$

"Convergence of the Eilenberg-Moore Spectral Sequence"

More general context:

N = finite CW complex

ΩN = based loop space

$$\Omega N \xrightarrow{ev_{1/2}} * \in N$$

$\leftarrow$ This is a fibration.

$\Big\}$ Fiber $\Omega_{**}N \times \Omega_{x*}N$

Use the Serre spectral sequence to compute

)

$$H_*(\Omega N) = H_*(N, C_*(\Omega_{*x} N) \otimes C_*(\Omega_{x*} N))$$

In the fiber over $*$, we have an inclusion

$$\Omega N \hookrightarrow \Omega N$$

with image consisting of paths constant on $[0, \frac{1}{2}]$.

This inclusion is a deformation retract via a retraction which preserves the length of loops.

Combine maps

$$\begin{tikzcd} C_*(\Omega N) \arrow[r, hook] & C_*(\Omega N) \arrow[l, bend left, dashed, "\substack{\text{retract + htpy} \\ \text{preserve length}}"] \arrow[r, "\simeq"] & C_*(N, C_*(\Omega_{*x} N) \otimes C_*(\Omega_{x*} N)) \arrow[l, bend left, "\text{retract + htpy}"] \end{tikzcd}$$

Preserve length up to error bounded by diam N with respect to the metric.

<u>Special Case:</u> $N = S^1$

Start w/ the Koszul resolution

$$0 \to \Gamma \otimes \Gamma \longrightarrow \Gamma \otimes \Gamma \longrightarrow \Gamma \longrightarrow 0$$

$$f \otimes g \longmapsto f \otimes g - zf \otimes z^{-1} g$$

$$f \otimes g \longmapsto fg$$

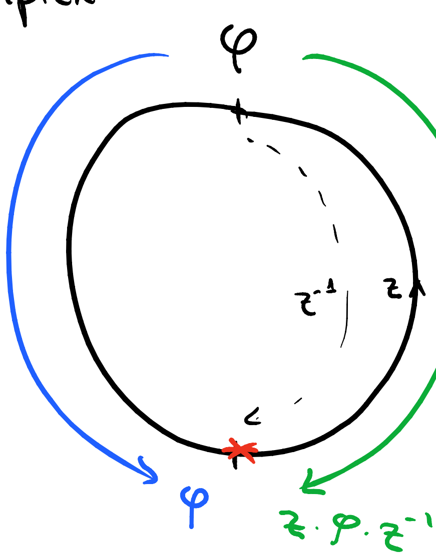
This is an acyclic complex!

Apply $\mathrm{Hom}_{\Gamma}(-, \Gamma)$

$$\begin{tikzcd} & & \Gamma \arrow[d, equal] \\ \mathrm{Hom}_{\Bbbk}(\Gamma, \Gamma) & \mathrm{Hom}_{\Bbbk}(\Gamma, \Gamma) \arrow[l] & \mathrm{Hom}_{\Bbbk}(\Gamma, \Gamma) \arrow[l] \end{tikzcd}$$

$$\varphi - z \varphi z^{-1} \longleftarrow\!\shortmid \varphi$$

"Morse complex"



Can write down explicit htpy of complex

$$C^*(S^1, \mathrm{Hom}(\mathcal{U},\mathcal{U})) \longleftarrow \Gamma$$

At level of

$$\begin{array}{ccccc} & & 1\otimes g & \longleftarrow\!\shortmid & g \\ \Gamma\otimes\Gamma & \longrightarrow & \Gamma\otimes\Gamma & \longrightarrow & \Gamma \end{array}$$

$$\left[\begin{array}{l} 0 = h(1\otimes g) \\ z^{i-1}\otimes zg + h(z^{i-1}\otimes zg) = h(z^{i}\otimes g) \end{array}\right.$$

$\underline{\underline{\text{e.g.}}}$

$$\left[\begin{array}{l} h(z\otimes g) = 1\otimes zg \\ h(z^{-1}\otimes g) + z^{-1}\otimes g = 0 \end{array}\right.$$

Given $P \subset \mathbb{R}^n \cong H^1(T^n;\mathbb{R})$,

we can define a completion Γ_P of Γ associated to the valuation

$$z^{\alpha} \longrightarrow \min_{p\in P} \langle \alpha, p\rangle$$

where

$$\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n \equiv \pi_1(\mathbb{T}^n)$$

$$p \in H^1(\mathbb{T}^n; \mathbb{R})$$

Take P = integral affine polytope

Consider "Koszul resolution"

$$\Gamma_p \otimes \Gamma_p \to \Gamma_p \otimes \Gamma_p \longrightarrow \Gamma_p$$

This is <u>not</u> exact, but it becomes exact upon completion:

$$\Gamma_p \hat{\otimes} \Gamma_p \to \Gamma_p \hat{\otimes} \Gamma_p \longrightarrow \Gamma_p$$

The h given above yields a nullhtpy of this completed complex!

i.e. h is bounded wrt the norm associated to this valuation.

Idea: If we unwind the inductive defn of h, terms look like

$$z^i \otimes g \longmapsto z^{i-j} \otimes z^{v} g$$

Norm of this is the ratio of $|z|^j$ in left factor and on right. These are equal!

(**Warning**: $\mathbb{R}_{discrete} \longrightarrow \mathbb{R}_{std}$

is acyclic complex which is not null-htpc!)

Why? There is a machinery for studying "derived category of top modules over top rings"

Problem: Γ_P is not "projective" as a Banach space

This computation can be generalized to show that for $P' \subset P$ an inclusion of polytopes, then there is an IM

$$\Gamma_P' \underset{\substack{\uparrow \\ \deg 0}}{\cong} H^*(\mathbb{T}^n, \mathrm{Hom}^c_\Lambda(\Gamma_P, \Gamma_P'))$$

$$H^*(\mathbb{T}^n, \mathrm{Hom}_\Lambda(\Gamma_p, \Gamma_{p'})) = \begin{cases} 0, & * \neq n \\ \underline{\mathrm{Hom}^c_\Lambda(\Gamma_p, \Gamma_p)}, & * = n \end{cases}$$

analogous

Note: $N = \text{mfld}$, $\bar{U} \subset V$

$$\mathrm{Hom}_{\text{sheaf}}(\mathbb{Z}_U, \mathbb{Z}_V) \cong \underline{H^*_c(U)}$$

If $P_0 \cap P_1 = \emptyset$,

then

$$H^*(\mathbb{T}^n, \mathrm{Hom}^c_\Lambda(\Gamma_{P_0}, \Gamma_{P_1})) = 0$$

One more essential computation: Tate acyclicity

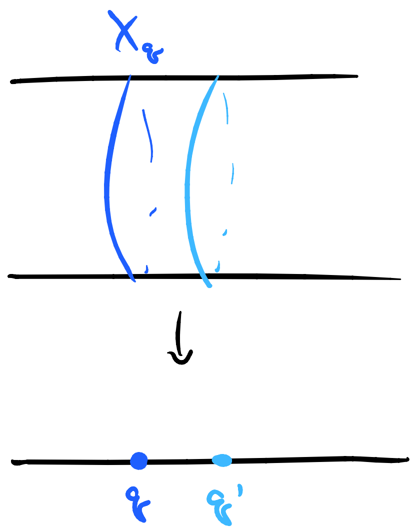
X symplectic

$$\Big\downarrow \text{ Lagrangian torus fibration}$$

$$Q$$

If $q \in Q$, then we have a nbhd canonically modelled after $H^1(X_q;\mathbb{R})$.

<u>Flux HM</u>



Morse area of cylinders with γ

$$X_q \longrightarrow X_{q'}$$

<u>Fukaya</u>: For each closed Lag'n L
(with well-defined HF^*)
which is transverse to X_q, there
exists $Q \supset P \underset{q}{\overset{\subset}{\ni}} H^1(X_q, \mathbb{R})$ s.t

$$HF^*(L, (X_q, \mathcal{U}_p)) \text{ is convergent}$$

$HF(L, (X_q, \mathcal{U}_P))$ is "convergent"

(see Jingyu's talk)

This is <u>not</u> invt under Hamn isotopy of L unless $P = \{0\} \subset H^1(X_q; \mathbb{R})$.

Idea for invariance:

Γ_P = ring of fns on Y_P

$Y_P = \{ y \in (\Lambda^*)^n \mid \mathrm{val}\, y \in P \}$

$Y_0 = \{ y \mid \mathrm{val}\, y = 0 \} < H^1(X_q; \mathbb{R})$

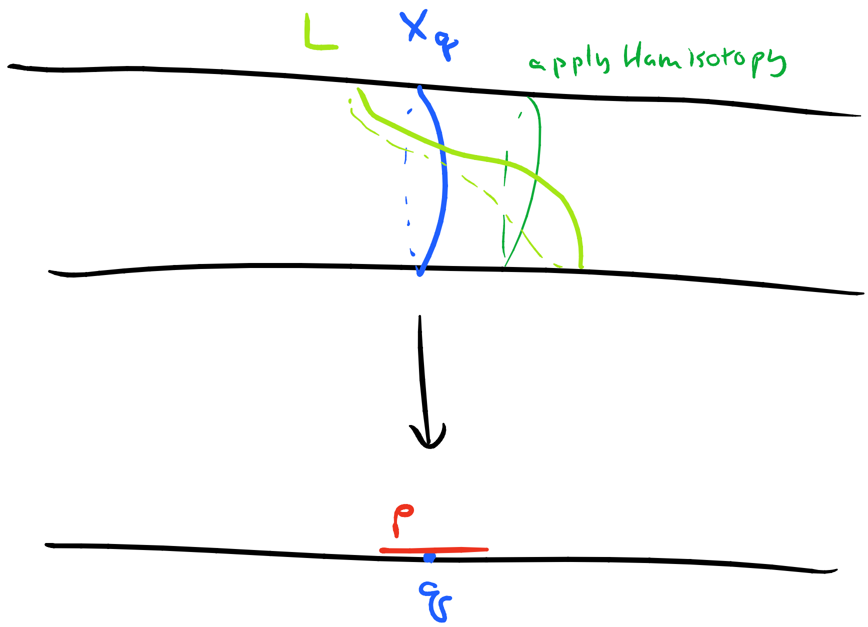
These are the "unitary local systems on X_q"

$a = a_0 +$ higher order in T

$a_0 \neq 0$

FOOO's work (+ Cho)

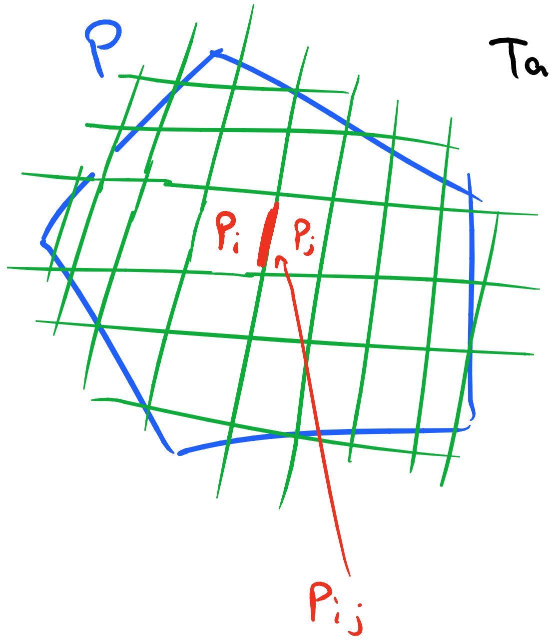
$\Longrightarrow HF^*(L, (X_q, \nabla_a))$ well-defined + invt.



$CF^*(L, (X_q, P))$ is non-zero as complex

Thm: $\forall\, L \subset X$ closed Lagn with well-defined HF^*, and $P \subset Q$ an integral affine polytope, there is a Floer homology group $HF^*(L, X_p)$ s.t. if P is suff small nbhd of q w/ $X_q \pitchfork L$, then

$$HF^*(L, X_p) \cong HF^*(L, (X_q, U_p))$$



Take cover of P, labelled by index set Σ

$\exists$ $HF^*(L, (X_q, \mathcal{U}_p))$ well-defined after deforming L by Ham isotopy.

Take $q_i \in P_i$

We also get, for $I \subset J \subset \Sigma$

$$HF^*(L, (X_{q_I}, \mathcal{U}_{p_I})) \longrightarrow HF^*(L, (X_{q_J}, \mathcal{U}_{p_J}))$$

$$(\Gamma_{p_I} \longrightarrow \Gamma_{p_J})$$

if the cover is sufficiently fine

Fukaya's method for defining the maps is to observe that if φ_j and φ_i are Ham'n defns of L that make it $\pitchfork$ X_{q_i} and X_{q_j}, we can assume that $\varphi_j(L) \pitchfork X_q$ for all $q \in P_k$ with $P_k \cap P_j \neq \emptyset$.

① $HF(L, (X_{q_i}, \mathcal{U}_{p_i})) \cong HF(L, (X_{q_i}, \mathcal{U}_{p_j}))$

② Over a single fibre we can use

$$\Gamma_i \longrightarrow \Gamma_{ij}$$

and continuation maps for HF^* from $\varphi_i L$ to $\varphi_j L$ to get the maps in the Čech complex.

$$HF^*(L, X_p) \equiv \check{H}^*(\underset{\substack{\uparrow \\ \text{cover}}}{\Sigma}, CF^*(L, (X_\bullet, \mathcal{U}_\bullet)))$$

Tate acyclicity:

$$\Gamma_p \longrightarrow \bigoplus_i \Gamma_{p_i} \longrightarrow \bigoplus_{i<j} \Gamma_{p_{ij}} \longrightarrow \cdots$$

is an acyclic complex

At the level of local systems, we get

$$\mathcal{U}_p \longrightarrow \bigoplus \mathcal{U}_{p_i} \rightrightarrows$$

complex over $\mathbb{T}^n$

complex over T^n

<u>Lemma</u>: Tate's Čech cpx admits a cts null-htpy.

$\Longrightarrow$ If all gps well-defined, then

$$HF^*(L,(x_p,U_p)) \longrightarrow \check{H}^*(\Sigma, \mathcal{F}(L,(x_\cdot,U_\cdot)))$$

is an IM