Denis Auroux: HMS - Cylinders, pants, and beyond - Part 1

Monday, July 4, 2016 8:57 AM

[Kontsevich 1994]:

$X^\vee$ mirror Calabi-Yau

$$\Longrightarrow \quad D^\pi \underbrace{\mathcal{F}(X)}_{\text{Fukaya category}} \simeq D^b\mathrm{Coh}(X^\vee)$$

and vice versa

(Baby example: T^2)

(1) In non-compact case,

$$\mathcal{F}(X) \longleftrightarrow \mathrm{Coh}_{\text{compact}}(X^\vee)$$

wrapped Fukaya category [Abouzaid-Seidel] $\mathcal{W}(X) \longleftrightarrow \mathrm{Coh}(X^\vee)$

(2) Non-CY case

$X \longleftrightarrow$ "Landau-Ginzburg model" $(X^\vee, W)$

(where $W \in \mathcal{O}(X^\vee)$)

superpotential

i.e.

$$X \xleftrightarrow{\text{HMS}} (X^\vee, W)$$

$$D\mathcal{F}(X,0) \longleftrightarrow D^b\mathrm{Sing}(W^{-1}(0)) = D^b\mathrm{Coh}(W^{-1}(0))\big/\mathrm{Perf}$$

unobstructed

λ: weakly obstructed

$W^{-1}(\lambda)$

Perfect complexes

$$D^b\mathrm{Coh}(X) \longleftrightarrow FS(X^\vee, W)$$

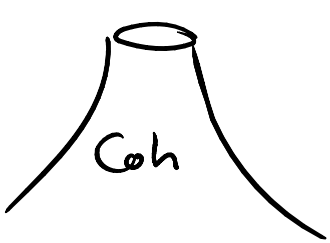
Fukaya-Seidel category

PLAN:

I. Cylinder

II. 

$\longleftrightarrow D^b_{\mathrm{Sing}}$

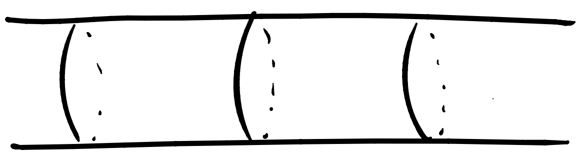
III. 

$\longleftrightarrow FS$

+ arbitrary hypersurfaces in $(\mathbb{C}^*)^n$ [Abouzaid-A.]

+ toric Fanos

CYLINDER

$X =$  $\longleftrightarrow$ $X^\vee = \mathbb{C}^*$ (or $\mathbb{K}^*$)

$$T^*S^1 = \mathbb{R}\times S^1$$

$$\omega = dr\wedge d\theta$$
$$= d(r\,d\theta)$$

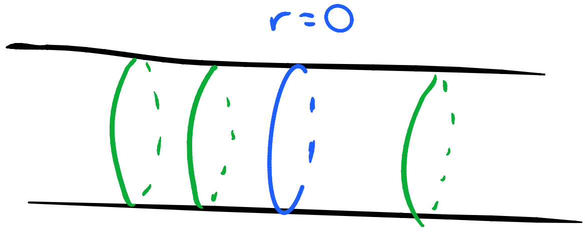
(0) $\mathcal{F}(X)$:

compact Lagn L

$$L \cong S^1$$

$$HF^*(L,L) \simeq H^*(S^1)$$

Only exact Lagn (up to Ham)
$r=0$



restrict to compact:
$\mathbb{R}$-worth of lagns (up to Ham)

$\cdots (L,L) = H^*(S^1)$ $\mathbb{R}$-worth of Lagns (up to Ham)

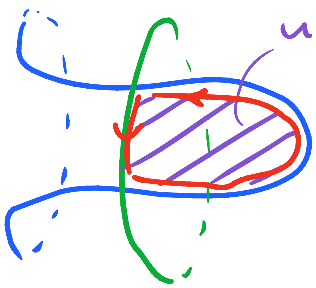
$HF^*(L,L') = 0$

Look at L + local systems

(L, ∇) with $\nabla =$ flat connection on trivial rank-1 bdle over L

$\mathrm{hol}(\nabla) \in \mathcal{U}(1)$

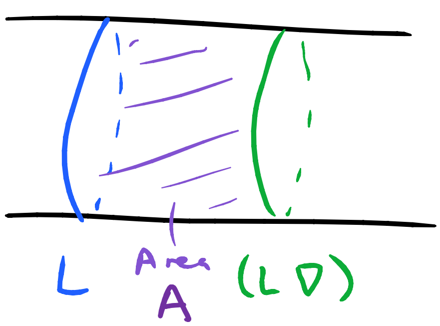
$\left(U_{\mathbb{K}} = \{ cT^0 + \ldots \mid c \neq 0 \} \right)$



Counts: $\pm T^{\int_{D^2} u^* \omega} \, \mathrm{hol}(\nabla|_{\partial u})$

True even if $(L,\nabla), (L,\nabla')$

$\{(L,\nabla)\} \cong \mathbb{K}^*$ objects, all orthogonal



$\longleftrightarrow$ $T^A \mathrm{hol}(\nabla) \in \mathbb{K}^*$

$e^{-A} \mathrm{hol} \in \mathbb{C}^*$

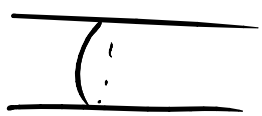
L_0 $\quad A \quad (L,\nabla)$ $\quad\quad \subset \mathbb{C}$

(family Floer theory, SYZ)

$$D\mathcal{F}(X) \simeq D^b Coh_{cpt}(\mathbb{K}^*)$$

NOT f.g. category

Each $(L,\nabla) \longleftrightarrow \mathcal{O}_p$, $p \in X^\vee$ skyscraper



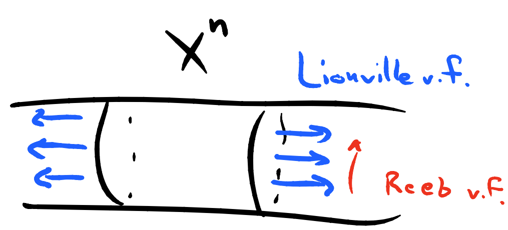
$$HF^*(\underset{L}{(L,\nabla)}, \underset{L'}{(L,\nabla)}) \simeq Ext^*(\underset{\mathcal{O}_p}{\mathcal{O}_p}, \underset{\mathcal{O}_p'}{\mathcal{O}_p}) \simeq \underset{\mathbb{C}}{H^*(S^1)}$$

(1) $\mathcal{W}(X)$ = wrapped Fukaya category [Abouzaid-Seidel]

X = Liouville mfld

= exact symplectic

$\omega = d\lambda$, $\lambda \overset{}{\underset{\omega}{\longleftrightarrow}} Z$ = Liouville v.f.



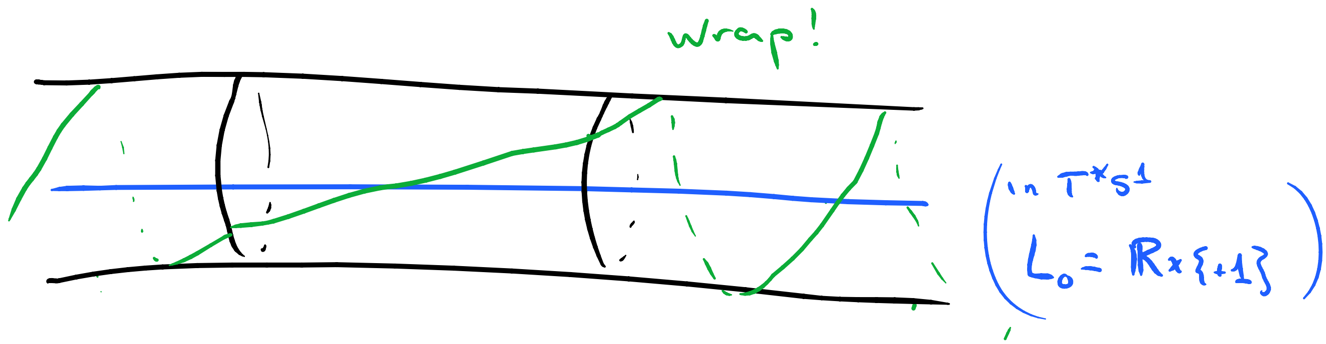
$(\partial X^n, \alpha = \lambda|_X)$
contact

ends $\left((1,+\infty)\times\partial X,\ d(r\alpha)\right)$

contact on ∂X

Allow: L exact Lagns, conical at ∞

$\mathbb{R}^+\times$ Legendrian



idea: Perturb Floer theory using Hamiltonian

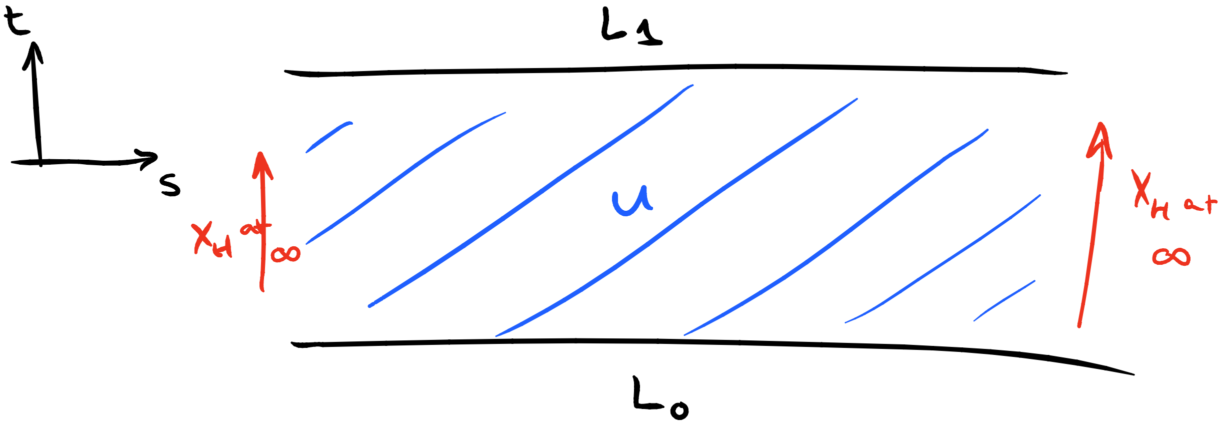
$$H=\frac{1}{2}r^2$$

$$\varphi_1^H(r,\theta)=(r,\theta+r)$$

$CW^\bullet(L_0,L_1):=$ generated by time-1 chords of X_H from L_0 to L_1

$$=\varphi_1^H(L_0)\cap L_1$$

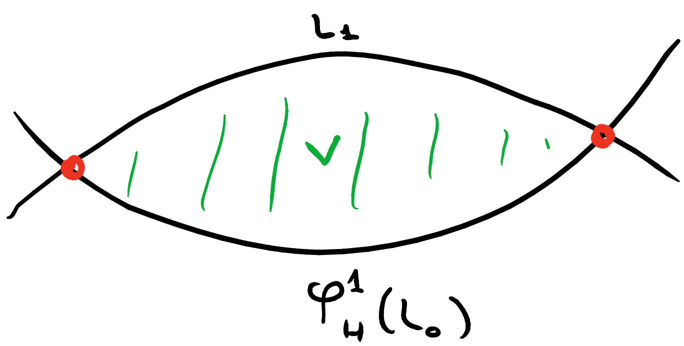
(= interior $\cap$'s and Reeb chords at contact ∂)



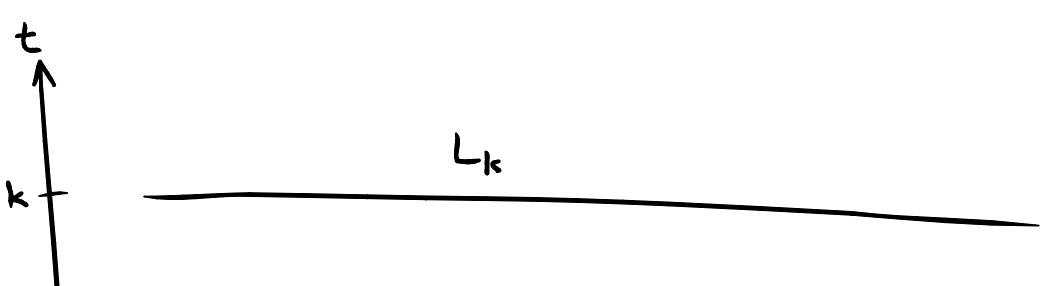
$$\frac{\partial u}{\partial s} + J\left(\frac{\partial u}{\partial t} - X_H\right) = 0$$

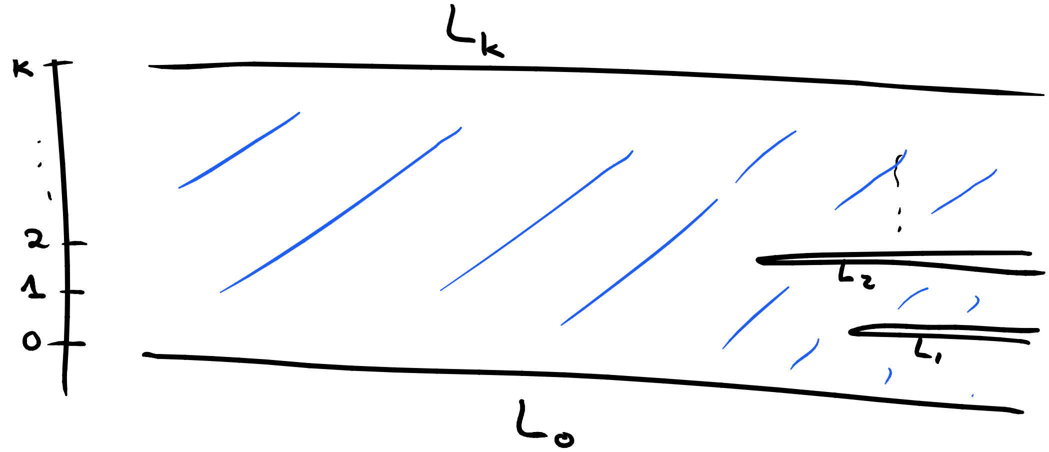
$$\Big\updownarrow \; v(s,t) = \varphi_{1-t}(u(s,t))$$

$$\frac{\partial v}{\partial s} + \tilde{J}_t \frac{\partial v}{\partial t} = 0$$



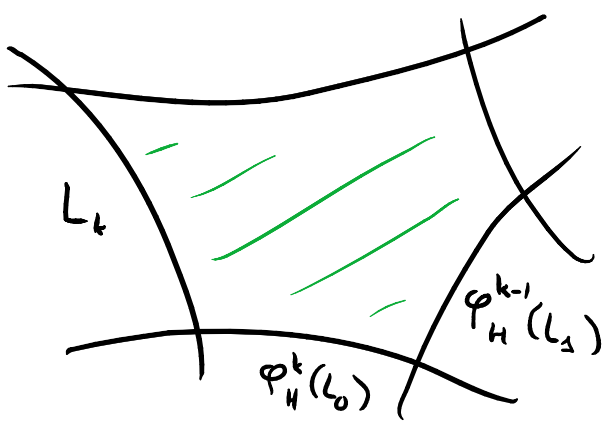
While...





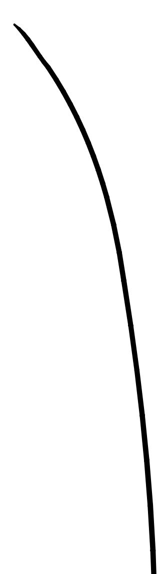
$\mathbb{R} \times [0,k] \setminus$ slits
$s \,,\, t$

same trick: $v(s,t) = \varphi^H_{k-t}(u(s,t))$

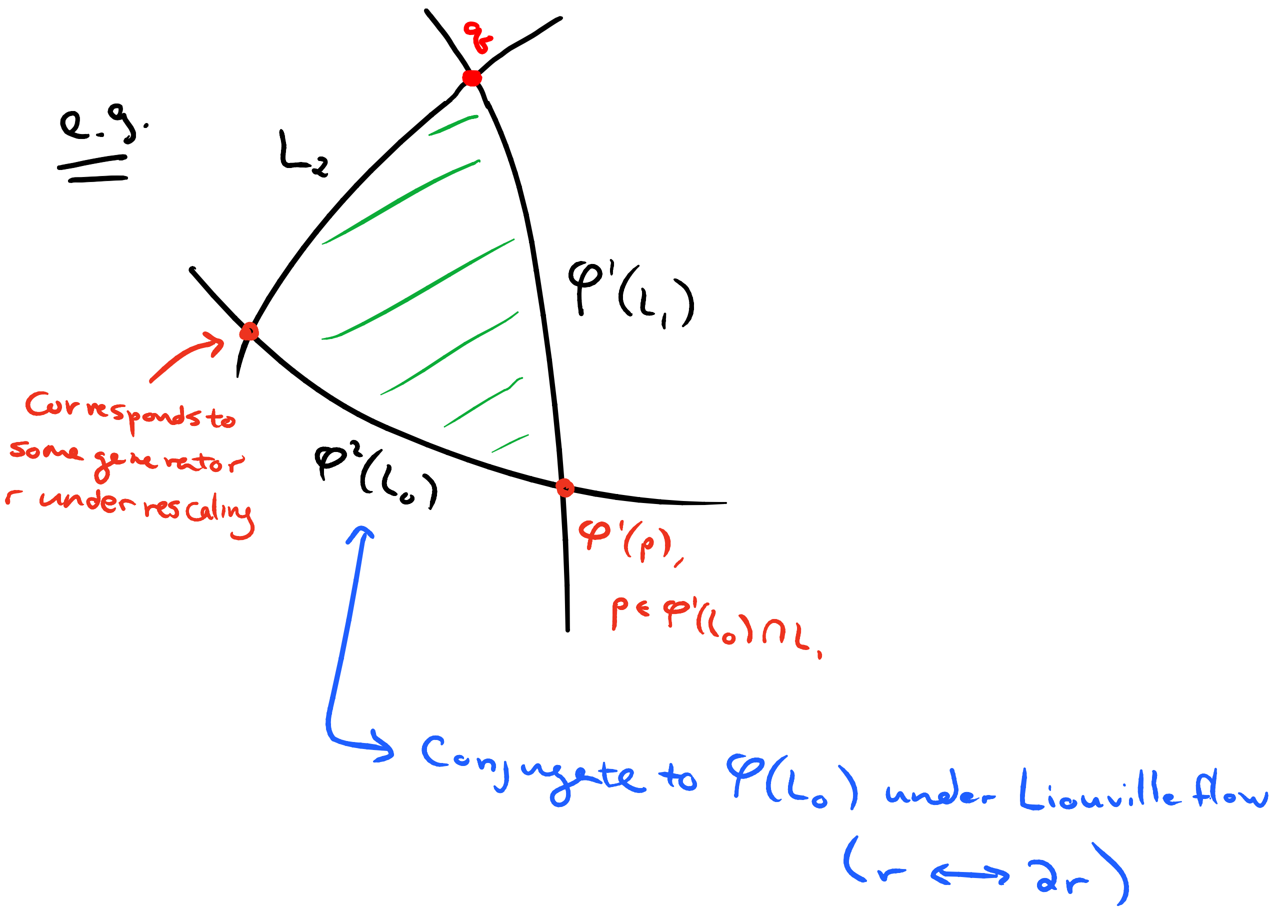


$\tilde{J}$-holo^c

e

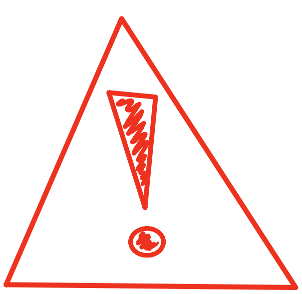


e.g.



$$\mu^k : CW^*(L_{k-1}, L_k) \otimes \cdots \otimes CW^*(L_0, L_1) \longrightarrow CW^*(L_0, L_k)$$

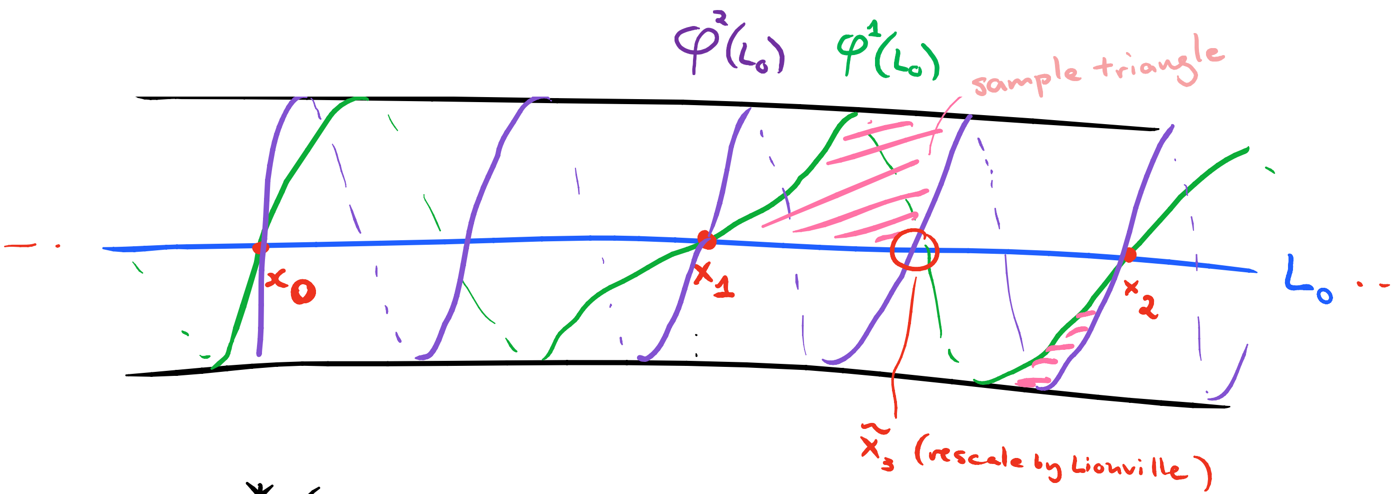
of degree $2-k$



Rescaling should be done more carefully

Let's compute for T^*S^1:

le

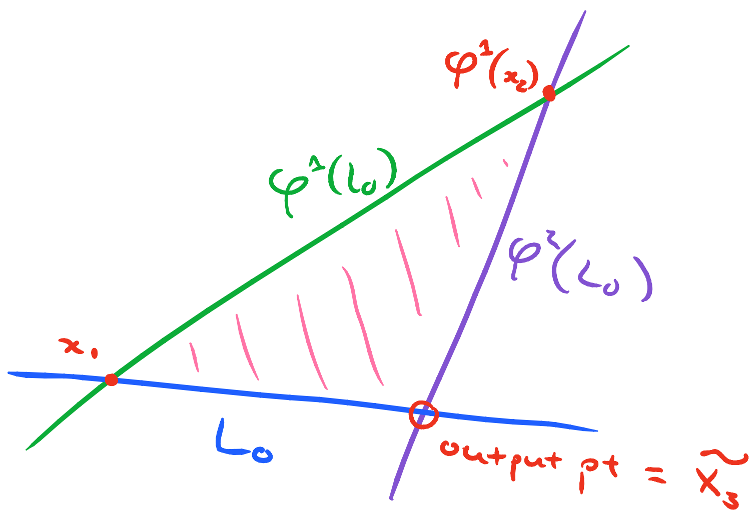


$$CW^*(L_0, L_0) = \mathrm{span}\{x_i,\ i \in \mathbb{Z}\}$$

all degree zero

$$\mu^1 = \text{diff'l} = 0$$

$$\mu^k = 0 \text{ for } k \geq 3$$



So $\mu^2(x_1, x_2) = x_3$ (no other contributing terms)

In fact, $\mu^2(x_i, x_j) = x_{i+j}$

$\rightsquigarrow$ call them x^i instead

$$\Longrightarrow \quad CW^*(L_0, L_0) \cong HW^*(L_0, L_0)$$
$$\cong \mathbb{K}[x^{\pm 1}]$$
$$\cong Ext^*(\mathcal{O}, \mathcal{O}) \text{ on } X^\vee$$

global functions

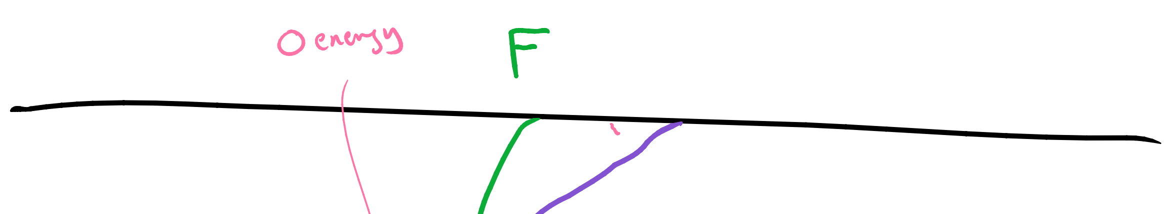
L_0 generates $\mathcal{W}(X)$ [Abouzaid] $+$ $\mathcal{O}$ generates $Coh(X^\vee)$

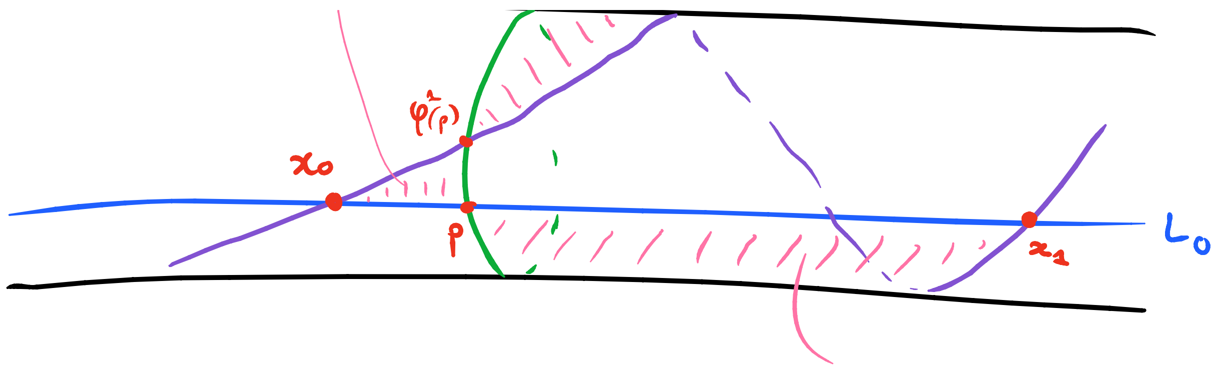
$$\Longrightarrow DW(X) \simeq D^b Coh(X^\vee)$$
$$\simeq \text{mod-}\mathbb{K}[x^{\pm 1}]$$

$$\underset{\text{Lagn in } X}{T} \longmapsto CW^*(L_0, T)$$

A_∞-module over $CW^*(L_0, L_0)$

$=$





$$CW(L_0, F) = \mathbb{K}$$

energy ξ = hol(F, ∇)

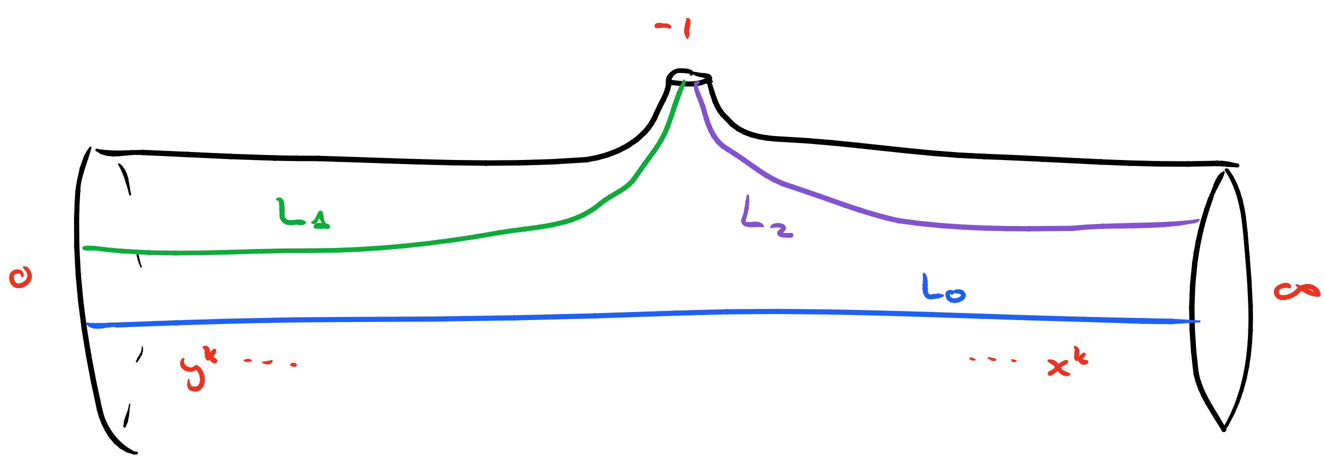
$CW(L_0, L_0)$: x acts by multiplication by some constant $\xi \in \mathbb{K}^\times$

= constant of the skyscraper corresp. to (F, ∇)

$\Longrightarrow$ as a module,

$$\{\mathcal{O} \xrightarrow{x-\xi} \mathcal{O}\} \qquad \mathbb{K}[x^{\pm 1}] \Big/ (x - \underbrace{\xi}_{\text{some constant}})$$

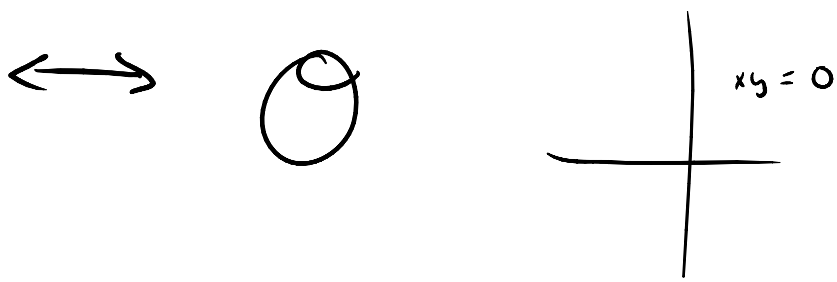
How about pair of pants?



Think of as $\mathbb{C}^* \setminus \{-1\} \cong (\mathbb{R} \times S^1) \setminus \{(0,\pi)\}$

Same triangles except some no longer exist:

$$CW^*(L_0, L_0) = \mathbb{K}[x,y] / xy = 0 \quad \begin{pmatrix} x = x_{+1} \\ y = y_{-1} \end{pmatrix}$$



We'll see how L_1 & L_2 fit into this picture tomorrow