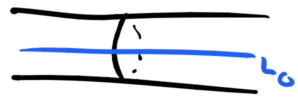
Yesterday:

HMS for $X =$  $\longleftrightarrow \quad X^\vee = \mathbb{C}^*$

$$\mathcal{W}(X) \quad \simeq \quad \mathrm{Coh}(X^\vee)$$

$$CW(L_0, L_0) \simeq \Bbbk[x^{\pm 1}]$$

In fact

$$T^*\mathbb{T}^n = (\mathbb{C}^*)^n \longleftrightarrow (\mathbb{C}^*)^n$$

$$L_0 = (\mathbb{R}_+)^n \longleftrightarrow \mathcal{O}$$

$$CW(L_0, L_0) = \Bbbk[x_i^{\pm 1}]$$

§ Compactification

$X \supset X^\circ$ open CY $\qquad X \setminus X^\circ = D$ divisor

(reduced, normal crossing)

$$[D] = c_1(X)$$

$\mathbb{C}P^1 - \mathbb{C}^* \cup \{0, \infty\}$

$\mathbb{CP}^1 = \mathbb{C}^* \cup \{0, \infty\}$

$\mathbb{C} = \mathbb{C}^* \cup \{0\}$

$\mathcal{F}^*(X)$ A_∞-deform^n of $\mathcal{F}(X^\circ)$ $\quad$ $\mathbb{Z}/2$ grading (includes $\mu = 2$ discs)

given by discs intersecting D

(disc hits D k times $\rightsquigarrow q^k$)

1st order deformation class (discs $\cap D = 1$)

class $\in HH^\circ(\mathcal{F}(X^\circ))$

$\longleftrightarrow \mathcal{O}(X^\vee)$ $\quad$ ($X^\vee$ = mirror of X°)

So mirror of X should be

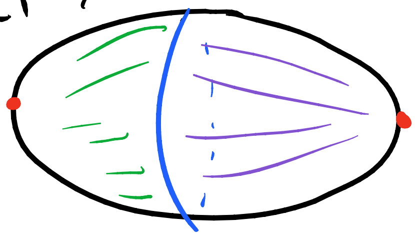
$$(X^\vee, W \in \mathcal{O}(X^\vee))$$

Landau-Ginzburg model

W encodes counts of holo^c discs hitting D once.

Example:

$\mathbb{CP}^1$, area $= A$



(F,∇) $\quad z = T^{\text{area}}\,\mathrm{hol}(\nabla)$

$\mathbb{C}^* \ni z$

$$W = T^{A/2}\left(z + z^{-1}\right) \text{ on } \mathbb{C}^*$$

$$\mathcal{F}(\mathbb{CP}^1) \longleftrightarrow D^b_{sing}(W)$$

equator, $\nabla_\pm$ $\qquad$ sky at 2 crit pts $z = \pm 1$

Also

$$D^b_{sing}(W) \underset{[\text{Orlov}]}{\simeq} \underline{\underline{MF}}(W)$$

matrix factorizations

$R = \mathbb{C}[z^{\pm 1}]$-modules

$$M_0 \underset{\partial}{\overset{\partial}{\rightleftarrows}} M_1$$

$$\partial^2 = \cancel{0}$$

$$\partial^2 = (\dot{W} - \text{const})\,\mathrm{id}$$

Reverse viewpoint

$\mathrm{Coh}(\mathbb{CP}^1)$ vs $\mathcal{FS}(\mathbb{C}^*, z + \frac{1}{z})$

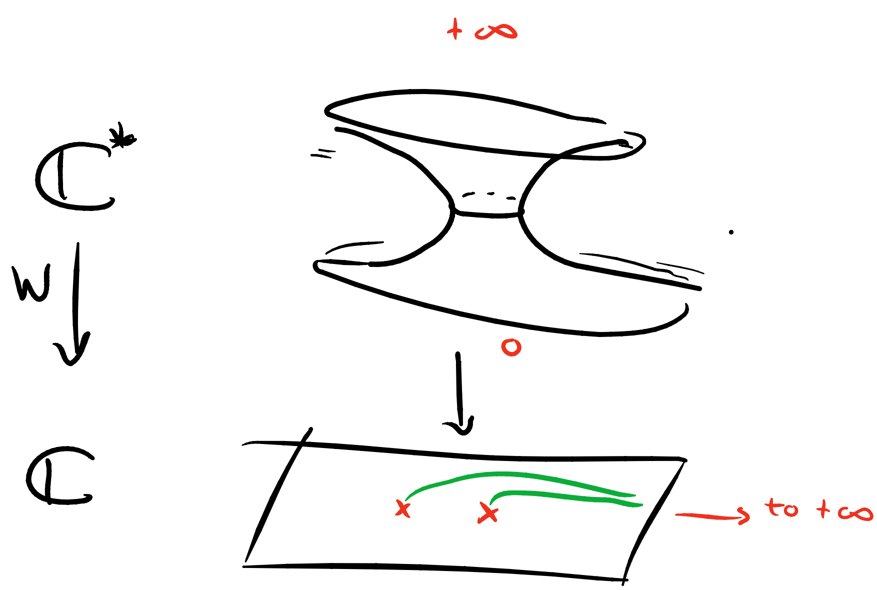
$$\mathcal{O} \rightrightarrows \mathcal{O}(1)$$

$\mathbb{C} \circlearrowright$ 2-dim $\circlearrowleft \mathbb{C}$

obj = Lagns L s.t.

$W|_L$ doesn't go towards $-\infty$

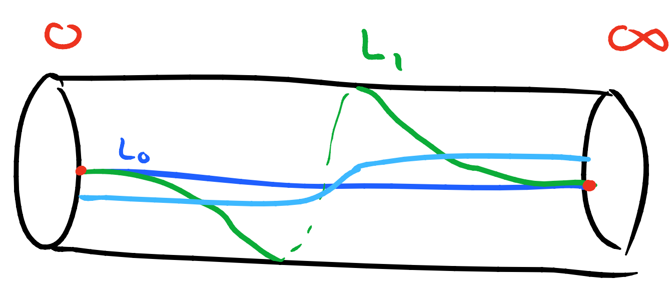
only goes towards $+\infty$



$CF(L_0, L_0) = \mathbb{k} \cdot \mathrm{id}$

$CF(L_0, L_1) = \mathrm{rk}\ 2$

$CF(L_1, L_0) = 0$



Can think of $CW(L_0, L_0) = \lim_{k\to\infty} CF(L_{-2k}, L_0)$

$$\operatorname{Hom}_{D^b(\mathbb{C}^*)}(\mathcal{O}, \mathcal{O}) = \mathbb{C}[z^{\pm 1}]$$

$$= \lim_{k\to\infty} \operatorname{Hom}_{D^b(\mathbb{P}^1)}(\mathcal{O}(-2k), \mathcal{O})$$

continuation
= mult^n by defining
section of D (in $\mathcal{O}(2)$)

allow $\leq k$ pole
order at $0, \infty$

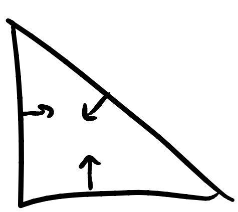
Similarly in higher dimensions,

$$X = \text{toric Fano} \supset X^0 = (\mathbb{C}^*)^n$$
(eg $\mathbb{CP}^n$)

Mirror

$X^\vee = (\mathbb{C}^*)^n$, $W =$ Laurent poly, 1 term for each ray of fan $\leftrightarrow$ exponents of monomials

e.g. $\mathbb{CP}^2$



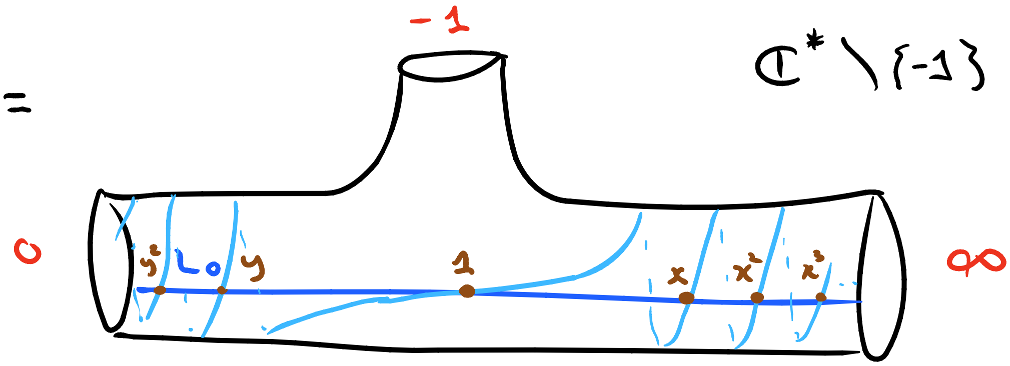
$$W = z_1 + z_2 + \frac{1}{z_1 z_2}$$

$\Big($ [Ab d] $\Big)$

$$\begin{pmatrix} \text{[Abouzaid]} \\ D^b\mathrm{Coh}(X) \simeq DFS(W) \end{pmatrix}$$

Back to Pair of Pants

$X =$

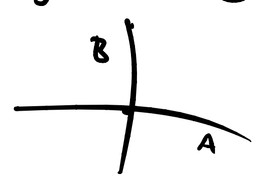


$\mathbb{C}^* \setminus \{-1\}$

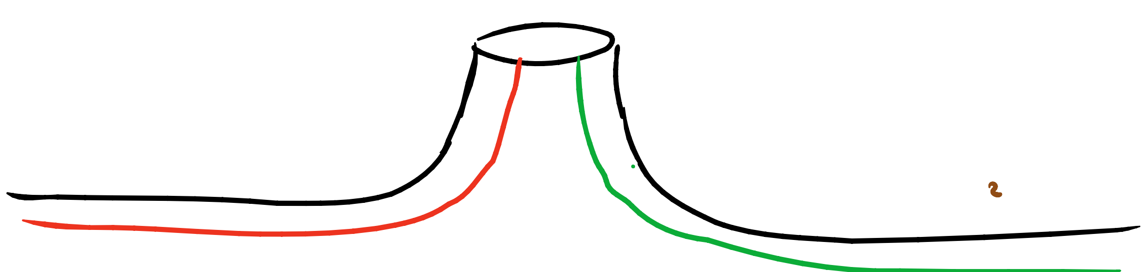
$$CW(L_0, L_0) \simeq \mathbb{C}[x,y] / xy=0$$

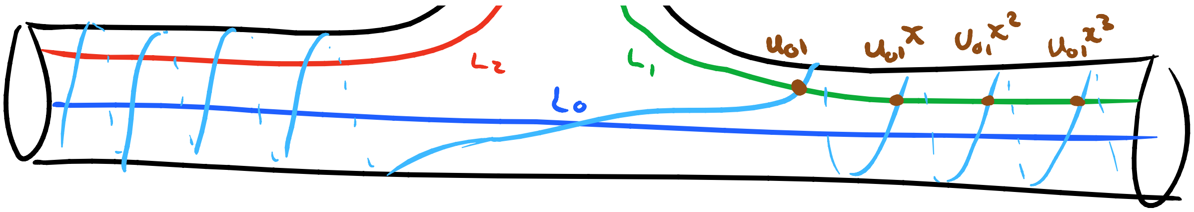
$$\simeq \mathrm{End}(\mathcal{O}_{X^\vee})$$

where $X^\vee = \{xy=0\} \subset \mathbb{C}^2$



But L_0 doesn't generate





$$CW(L_0, L_0) \simeq \mathbb{C}[x,y]/_{xy=0}$$

$$\begin{tikzcd}
& L_0 \arrow[loop above] \arrow[dl, shift right, "{\cancel{u_{01}}\,\mathbb{C}[x]}"', "{x^{1/2}}"] \arrow[dr, shift left, "{\cancel{u_{02}}\,\mathbb{C}[y]}", "{y^{1/2}}"'] & \\
\mathbb{C}[x,z]/_{xz=0} \circlearrowright L_1 \arrow[ur, shift right, "{\cancel{u_{10}}\,\mathbb{C}[x]}"', "{x^{1/2}}"] \arrow[rr, shift left, "{\cancel{u_{12}}\,\mathbb{C}[z]}", "{z^{1/2}}"'] & & L_2 \circlearrowleft \mathbb{C}[y,z]/_{yz=0} \arrow[ul, shift left, "{\cancel{u_{20}}\,\mathbb{C}[y]}", "{y^{1/2}}"'] \arrow[ll, shift left, "{\cancel{u_{21}}\,\mathbb{C}[z]}", "{z^{1/2}}"']
\end{tikzcd}$$

Also

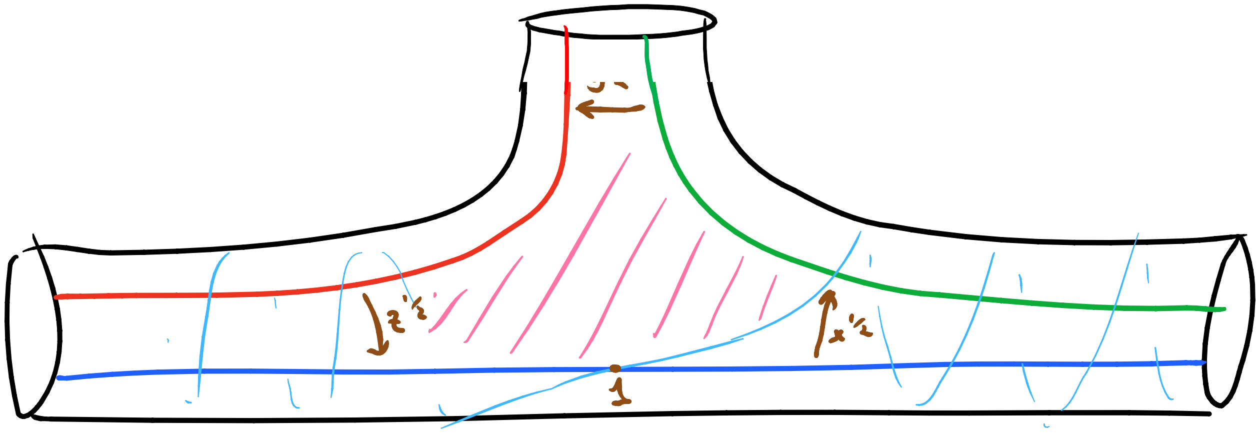
$$\mu^3(x^{1/2}, y^{1/2}, z^{1/2}) = -\mathrm{id}$$

$$\mu^3(x^{1/2}, z^{1/2}, y^{1/2}) = -\mathrm{id}$$

Exact triangles

$$\begin{tikzcd}
L_0 \arrow[r] & L_1 \arrow[dl] \\
L_2 \arrow[u] &
\end{tikzcd}
\qquad
\begin{tikzcd}
L_0 \arrow[r] & L_2 \arrow[dl] \\
L_1 \arrow[u] &
\end{tikzcd}$$

+ cyclic permutations



HH^* calculation $\Longrightarrow$ this determines A_∞-structure

[see AAEKO]
Abouzaid Auroux Efimov Katzarkov Orlov

$\mathcal{O}$, $\quad \mathcal{O}_A$, $\quad \mathcal{O}_B$

$R = \mathbb{C}[x,y]/xy \qquad R/y \qquad R/x$

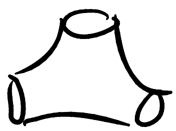
$$\begin{tikzcd}
 & R \arrow[loop right, "{\mathbb{C}[x,y]/xy}"] \arrow[dl, shift right, "{\mathbb{C}[x]}"'] \arrow[dr, shift left, "{\mathbb{C}[y]}"] & \\
R/y \arrow[loop left, "{\mathbb{C}[x,z]/xz=0}"] \arrow[ur, shift right, "{x\mathbb{C}[x]}"'] \arrow[rr, shift left, "{z^{1/2}\mathbb{C}[z]}"] & & R/x \arrow[loop right, "{\mathbb{C}[y,z]/yz=0}"] \arrow[ul, shift left, "{y\mathbb{C}[y]}"] \arrow[ll, shift left, "{z^{1/2}\mathbb{C}[z]}"]
\end{tikzcd}$$

$$z \in \mathrm{Ext}^2(R/y, R/y) \xleftarrow{\; z^{1/2}\mathbb{C}[z] \;} \quad R/x \qquad yz=0$$

$$x\,\mathbb{C}[x] \to \mathbb{C}[x,y]/_{xy=0} \to \mathbb{C}[y]$$

<u>Thm</u> [AAEKO]:

$$W\Big(\ \Big) \simeq \mathrm{Coh}(X)$$

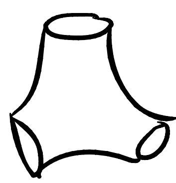


Knörrer Periodicity

<u>Thm</u> [Orlov]: Equivalent, via stabilization

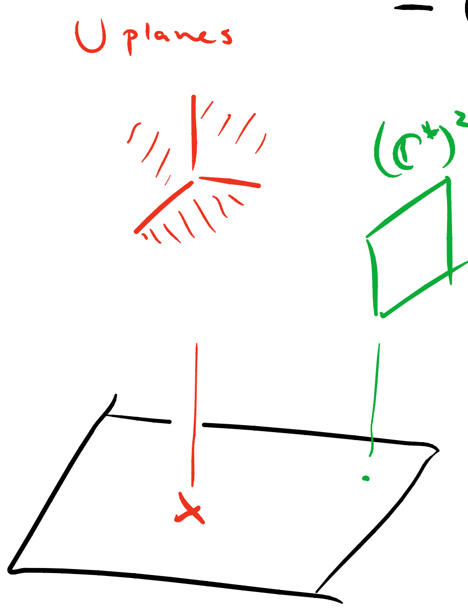
<u>Thm</u>:

$$W\Big(\ \Big) \simeq D^b_{Sing}(\mathbb{C}^3, -xyz)$$



$$= \mathrm{Coh}(xyz)/\mathrm{Perf}$$

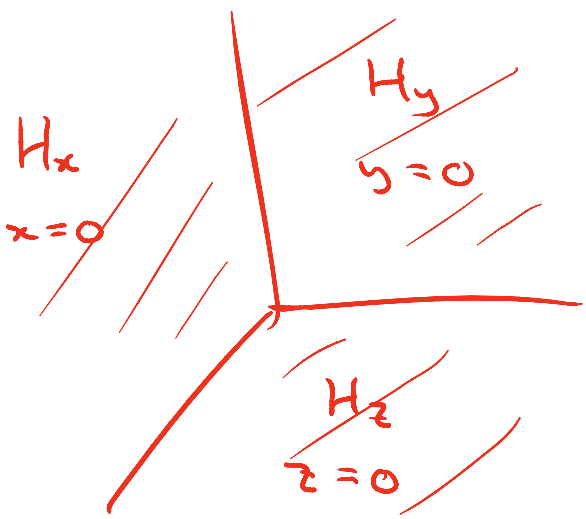
$$\mathbb{C}^3 \xrightarrow{xyz} \mathbb{C}$$



Thm [Orlov]

$$D^b_{sing}(W) \simeq MF(W) \cong D^b\mathrm{Coh}\{xy=0\}$$

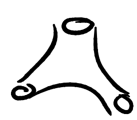
$$:= D^b\mathrm{Coh}(W^{-1}(0)) / \mathrm{Perf}$$

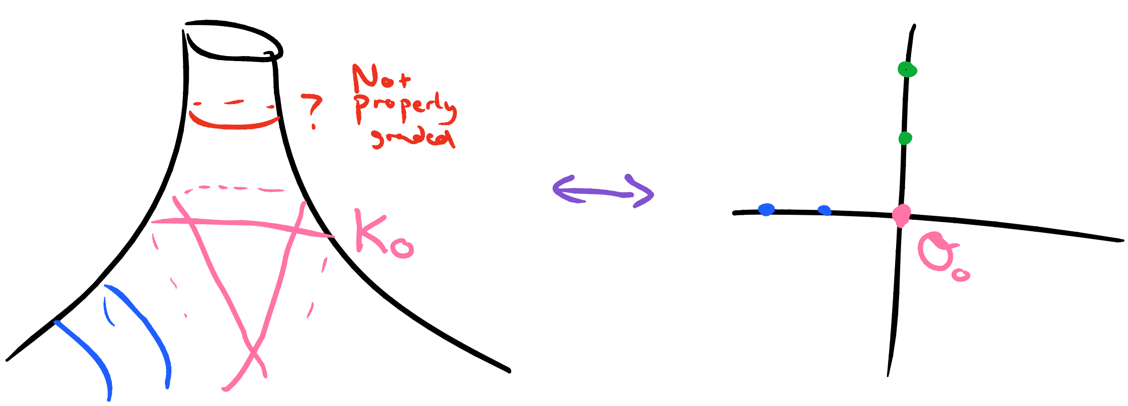


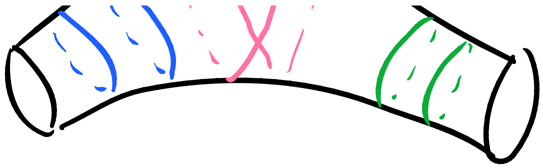
$$\hat{R} \underset{yz}{\overset{x}{\rightleftarrows}} \hat{R} \qquad \mathcal{O}_{(x=0)}$$

$$\hat{R} \underset{xz}{\overset{y}{\rightleftarrows}} \hat{R} \qquad \mathcal{O}_{(y=0)}$$

$$\hat{R} \underset{xy}{\overset{z}{\rightleftarrows}} \hat{R} \qquad \mathcal{O}$$

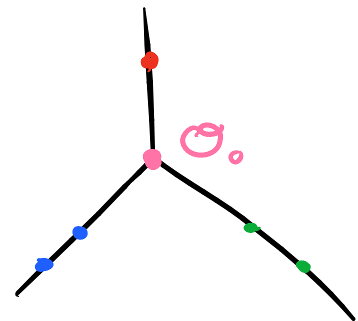
Compact Lagns on 





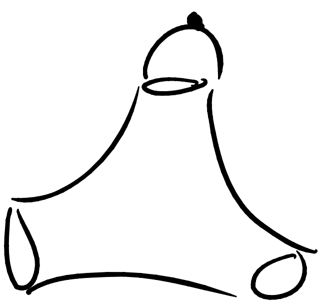
OR

$\underline{\mathbb{C}^3, -xyz}$:



Compactify:

$\mathbb{C}^* = \text{pants} \cup \{-1\}$



$\longleftrightarrow$

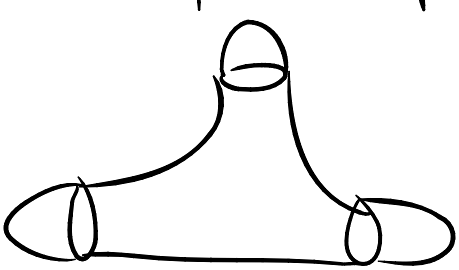
$$\left(\mathbb{C}^3,\ W = -xyz + qz\right)$$
$$= z(q - xy)$$

Morse-Bott along

$\{z = 0,\ xy = q\} \simeq \mathbb{C}^*$

Stabilization of $\mathbb{C}^*$

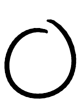
$\mathbb{C}P^1 = \text{pants} \cup \{3 \text{ pts}\}$



$\longleftrightarrow$

$$\left(\mathbb{C}^3,\ W = -xyz + q(x+y+z)\right)$$

Stabilization of usual mirror of $\mathbb{C}P^1$



Orbifold Compactifications

$$\longleftrightarrow \left(\mathbb{C}^3, W = -xyz + \varepsilon(x^k+y^k+z^k)\right)$$

$$\uparrow \mathbb{Z}/k, \ k = 2g+1$$

lifts

Genus g curve

$$\longleftrightarrow \overbrace{(\mathbb{C}^3, W)/\mathbb{Z}/k}^{\text{crepant resolution}}$$

Next:

* **Higher dim pants:**

$$\Pi_n = \mathbb{CP}^n \setminus (n+2) \text{ hyperplanes}$$
$$= (\mathbb{C}^*)^n \setminus \{\textstyle\sum x_i = -1\}$$

$W($

$$W(\Pi_n) \simeq \mathrm{Coh}\left(\{x_1 \cdots x_{n+1} = 0\} \subset \mathbb{C}^{n+1}\right)$$

$$\simeq D^b_{Sing}\left(\mathbb{C}^{n+2}, -\prod x_i\right)$$

not yet fully calculated

$$\Big\downarrow /\langle \mathcal{O} \rangle$$

$$D^b_{Sing}\Big($$  $$\Big)$$

$$W(\Pi_n) \longrightarrow W(\Pi_{n-1})$$

$$W(\Pi_n) \simeq (\mathbb{C}^*)^n \setminus \Pi_{n-1}$$