Derived symplectic geometry and applications
Damien Calaque

I. Symplectic linear algebra in the derived setting
stable ∞-categories, dg-categories, A_∞-categories

II. Shifted symplectic and Lagrangian ~~constructi~~ structures on derived stacks.
diff. calculus on derived stacks
examples

III. An approach to topological Fukaya categories of surfaces
[Dyckerhoff-Kapranov Brav-Dyckerhoff]

I. Symp. linear alg. in the derived setting

$\mathcal{A}$ abelian k-linear category k = field of char 0.
$Cpx(\mathcal{A})$ = cochain cpx in $\mathcal{A}$.

- $D(\mathcal{A})$ = localization of $Cpx(\mathcal{A})$ w.r.t. quasi-iso's.
  - $D(\mathcal{A})$ is a triangulated category

$$\underset{[1]}{\overset{}{\Sigma}} : D(\mathcal{A}) \to D(\mathcal{A})$$

$\Sigma = [1]$

- $D(\mathcal{A})$ is the homotopy category of an ∞-category.
  - Various models for ∞-categories:
    - topological/simplicial categories
    - Segal categories (Simpson, Toen-Vezzosi)
    - Complete Segal spaces (C.S.S.) (Rezk)
    - quasi-categories (Joyal, Lurie)

Main feature: in an ∞-category, given two objects x and y there is a space of morphisms $\mathrm{Hom}(x,y)$.

The <u>homotopy category</u> $h(\mathcal{C})$ of an ∞-category $\mathcal{C}$ is defined as follows: objects are objects of $\mathcal{C}$

$$\mathrm{Hom}_{h(\mathcal{C})}(x,y) := \pi_0\big(\mathrm{HOM}(x,y)\big)$$

<u>Definition</u>: an ∞-category $\mathcal{C}$ is <u>stable</u> if:

* it has a 0 object (namely an object which is both initial and terminal)
* it admits pullbacks
* the loop functor $\Omega : \mathcal{C} \to \mathcal{C}$ defined by the pullback diagram

$$\begin{tikzcd} \Omega X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & 0 \arrow[d] \\ 0 \arrow[r] & X \end{tikzcd}$$

is an <u>equivalence</u>.

<u>Example</u>: $\mathcal{C} = C_{px}(k\text{-mod}) \qquad \Omega = [-1]$

pullbacks are given by mapping cones.

<u>Remarks</u>

* push-outs do exist as well
* the inverse of Ω is $\Sigma : \mathcal{C} \to \mathcal{C}$

$$\begin{tikzcd} \Sigma X & 0 \arrow[l] \\ 0 \arrow[u] \arrow[ur, phantom, "\lrcorner", very near end] & X \arrow[l] \arrow[u] \end{tikzcd}$$

<u>Properties</u>: if $\mathcal{C}$ is a stable ∞-category then $h(\mathcal{C})$ is triangulated.

* The shift functor is exactly the suspension functor
* distinguished triangles $X \to Y \to Z \to \Sigma X$

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ 0 \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & Z \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r] \arrow[d] \arrow[rr, dashed, bend left, "0"] & Y \arrow[r] & Z \arrow[d] \\ 0 \arrow[rr] & & \Sigma X \end{tikzcd}$$

- $\mathrm{Cpx}_\infty(\mathcal{A}) = \infty$-categorical localization of $\mathrm{Cpx}(\mathcal{A})$ at quasi-isos

<u>Claim</u>: $\mathrm{Cpx}_\infty(\mathcal{A})$ is stable.

<u>Idea of proof</u>:
- 0 object ✓
- existence of pushouts:

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "f"'] & Z \arrow[d, dashed] \\ Y \arrow[r, dashed] & \exists \end{tikzcd}$$

$$\Leftrightarrow \quad \begin{tikzcd} X \arrow[r] \arrow[d, "f\,g"'] & 0 \arrow[d] \\ Y \oplus Z \arrow[r, dashed] & \exists \end{tikzcd}$$

$\exists$ ⇝ mapping cone of the morphism $f-g : X \to Y \oplus Z$

<u>Remarks</u>. * $h(\mathrm{Cpx}_\infty(\mathcal{A})) = D(\mathcal{A})$

One feature of $\mathcal{A}$ that we did not use: k-linearity.
$\mathcal{A}$ is k-linear $\Rightarrow \mathrm{Cpx}_\infty(\mathcal{A})$ is k-linear.
($\mathrm{HOM}(x,y)$ = simplicial k-mod).

Model for k-linear ∞-cat: categories enriched in ⇝ simp. k-mod
⇝ coch. cpx ≤ 0

<u>Look for a model for stable ∞-categories</u>
A first attempt would be dg-categories. This does not work for stable cats, we add more data:

<u>Definition</u>:
- let $\mathcal{T}$ be a dg-category
- a left $\mathcal{T}$-module is a functor $F : \mathcal{T} \to \mathrm{Cpx}(k\text{-mod})$
- a right " , " " $\mathcal{T}^{op}$ "
- write $\hat{\mathcal{T}}$ = cat. of right $\mathcal{T}$-modules
$\hat{\mathcal{T}}$ is a dg category.

We have a functor $\mathcal{T} \longrightarrow \hat{\mathcal{T}}$

$x \longmapsto \mathrm{Hom}(-,x)$

<u>Definition</u>: $\mathcal{T}$ a d.g. category is called (strongly) <u>pretriangulated</u> if:

- $\exists\, 0 \in \mathcal{T}$ such that $\mathrm{Hom}(-,0) \cong \underline{0}$
- $\forall\, M \in \hat{\mathcal{T}}$ such that $\Sigma M \cong \mathrm{Hom}(-,c)$ for some c then $\exists\, d \in \mathcal{T}$ such that $M \cong \mathrm{Hom}(-,d)$
- $\forall\, F: \mathrm{Hom}(-,c) \to \mathrm{Hom}(-,d)$ the (object wise) mapping cone of F is representable as well.

Let $\mathcal{T}$ be a dg category:

* $h(\mathcal{T}) :=$ same objects as $\mathcal{T}$

  $\mathrm{Hom}_{h(\mathcal{T})}(x,y) = H^0(\mathrm{Hom}(x,y))$

* $h(\mathcal{T})$ is triangulated if $\mathcal{T}$ is pre-triangulated
* $\mathrm{tria}(\mathcal{T}) =$ the thick subcategory of $h(\hat{\mathcal{T}})$ generated by $h(\mathcal{T})$ as a triang. category

* $\mathrm{tria}(\mathcal{T})$ has a dg-enhancement, $\mathrm{tw}(\mathcal{T})$.

  <u>Construction of $\mathrm{tw}(\mathcal{T})$</u> (the "pre-triang. cat of twisted cpx's")

  1st step: $\mathbb{Z}\mathcal{T}$ made of $A[n]$ $\quad A \in \mathcal{T}$ $\quad n \in \mathbb{Z}$

  $\mathrm{Hom}(A[n], B[m]) = \mathrm{Hom}(A,B)[m-n]$

  2nd step: objects of $\mathrm{tw}\,\mathcal{T}$ $\quad (A_1[n_1], \dots, A_k[n_k], \delta)$

  where $\delta_{ij} \in \mathrm{Hom}^1(A_i[n_i], A_j[n_j])$

  such that $\delta_{ij} = 0 \quad i \geq j$

  $d\delta + \delta^2 = 0.$

  δ is a matrix of elements in Hom cpx's.

<u>A_∞-categories</u> equivalent (more flexible) to dg-cat.

<u>definition</u>: * a collection of objects

* $\forall\, x, y$ objects $Hom(x,y)$ is a $\mathbb{Z}$-graded k-mod.

* $\forall\, n \geq 1$, $\forall\, x_0, \dots, x_n$ objects we have

$$m_n : Hom(x_{n-1}, x_n) \otimes \dots \otimes Hom(x_0, x_1) \longrightarrow Hom(x_0, x_n)$$

a map of degree $2-n$. and

these satisfy $\sum_{r+s+t=n} (-1)^{r+st}\, m_{r+1+t}\left(id^{\otimes r} \otimes m_s \otimes id^{\otimes t}\right) = 0$

- for instance $m_1 \circ m_1 = 0 \Rightarrow$ Hom are complexes
- $m_1 \circ m_2 = m_2 \circ (m_1 \otimes id + id \otimes m_1) \Rightarrow m_2$ is a morphism of complexes
- $m_2(m_2 \otimes id - id \otimes m_2) = [m_1, m_3]$

$\Rightarrow$ m_2 is assoc. up to homotopy.

T A_∞-cat $\Rightarrow h(T)$ homot category.

<u>Subtlety</u>: dealing with units.

<u>Assume</u> $\exists\ 1_x \in Hom^\circ(x,x)$ s.t. $m_1(1_x) = 0$
($\forall$ objects x)

$$m_2(1_x, f) = m_2(f, 1_x) = f$$

and $m_n(\dots, 1_x, \dots) = 0 \quad n \geq 3$

This is called a strict unit.

A_∞ functor : $F_n : Hom(x_{n-1}, x_n) \otimes \dots \otimes Hom(x_0, x_1)$

$$\downarrow$$

$$Hom(F(x_0), F(x_n))\ [1-n]$$

satisfying identities.

$\hat{T}$ = cat of A_∞ functors $T^{op} \to Cpx(k\text{-mod})$

References • B. Keller
• Lee Cohn.

Damien Calaque II

Symplectic linear algebra in the ∞-cat. setting

$\mathcal{C} = Cpx_\infty(k\text{-mod})$ = stable ∞-cat of complexes of k-mods.

Note: works for $\mathcal{C}$ stable symm. monoidal ∞-cat.

V object in $\mathcal{C}$. Assume V is perfect, so there exists $V^\vee$ s.t. $Hom(V \otimes W, Z) \simeq Hom(W, V^\vee \otimes Z)$

Definition: an n-shifted symplectic structure on V is a pairing $\omega : V \otimes V \to 1\!\!1[n]$ such that
- ω factors through $\Lambda^2 V := S^2(V[-1])[2]$
- ω is non deg. meaning that the induced morphism $V \to V^\vee[n]$ is a quasi-iso.

Examples 1) M closed oriented manifold of dim n, connected.

$W = \Omega^*(M, k)$

$$W \otimes W \xrightarrow{\wedge} W \xrightarrow{\text{project}} \frac{\Omega^*(M,k)[-n]}{d(\Omega^{n-1}(M,k))} \xrightarrow{\int_M} k[-n]$$

This pairing is symmetric.

On $V := W[1]$ the pairing ~~becomes~~ becomes a $(2-n)$-shifted symplectic pairing.

2) X smooth projective CY-variety of dim n. / X is a cpx manifold compact, CY, dim n

$W = R\Gamma(\mathcal{O}_X)$ / $W = \mathcal{A}^{0,*}(X, \mathbb{C})$, $d = \bar\partial$

Serre duality tells you there is a $(2-n)$-symplectic structure on $V = W[1]$.

Assume that (V, ω) is an n-symplectic cpx.

$L \xrightarrow{f} V$ morphism. L perfect as well.

<u>definition</u> • an <u>isotropic structure</u> on f is a path between $f^*\omega$ and 0 in the space $\mathrm{Hom}(\wedge^2 L, \mathbb{k}[n])$.

• an isotropic structure on f (or L) is ND (or Lagrangian)

if

$$\begin{tikzcd} & L^0 \arrow[r, "\sim"] \arrow[d] & L^\perp \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & 0 \arrow[d] \\ L \arrow[ur, dashed, "\sim"] \arrow[r, "f"'] & V \arrow[r, "\sim"] & V^\vee[n] \arrow[r] & L^\vee[n] \end{tikzcd}$$

<u>Example</u> 1) $V = \Omega^*(M, \mathbb{k})[1]$ M closed oriented mfld.

$M = \partial N$ N oriented as well.

$L = \Omega^*(N, \mathbb{k})_{[1]} \xrightarrow{i^*} V$ $i : M \to N$

There is a Lagrangian structure on $f = i^*$

The homotopy between $f^*\omega$ and 0 is given by $\int_N$ (Stokes' formula)

2) $V = \mathbb{R}\Gamma(\mathcal{O}_X)$ X n-dim CY

$X \hookrightarrow Y^{n+1}$ $[X] = -K_Y$

$L = \mathbb{R}\Gamma(\mathcal{O}_Y)[1] \longrightarrow V$ has a Lagrangian structure.

(relative Serre duality).

Symplectic geometry with derived stacks

$\mathrm{Aff} = (\mathrm{alg}/k)^{op}$

$$\mathrm{Aff}^{op} \longrightarrow \mathrm{Set}$$
$$\searrow \mathrm{Gps}$$

alg spaces: $\mathcal{U} = \bigcup(\mathcal{U}_i)$ open cover

$$F(\mathcal{U}) \xrightarrow{\sim} \lim \left(\prod F(\mathcal{U}_i) \rightrightarrows \prod_{jk} (\mathcal{U}_j \cap \mathcal{U}_k) \right)$$

stacks

$$F(\mathcal{U}) \xrightarrow{\sim} 2\text{-}\lim \left(\begin{array}{c} \prod F(\mathcal{U}_i) \\ \downarrow\downarrow \\ \prod F(\mathcal{U}_j \cap \mathcal{U}_k) \\ \downarrow\downarrow\downarrow \\ \prod F(\mathcal{U}_a \cap \mathcal{U}_b \cap \mathcal{U}_c) \end{array} \right)$$

higher stacks

$F: \mathrm{Aff}^{op} \to \mathrm{Spaces}$ such that

$$F(\mathcal{U}) \longrightarrow \lim \left(\prod_i F(\mathcal{U}_i) \rightrightarrows \prod_{jk} F(\mathcal{U}_j \cap \mathcal{U}_k) \rightrightarrows \cdots \right)$$

if your space is the classifying space of a grupoid then this definition boils down to the ~~[illegible]~~ usual definition of a stack.

Replace Aff by $d\mathrm{Aff} = (\mathrm{cdga}^{\leq 0}_{\infty})^{op}$

$\mathrm{cdga}^{\leq 0}$ = category of commutative diff. graded algebras$/k$ concentrated in ≤ 0 degree.

$\mathrm{cdga}^{\leq 0}_{\infty} = \mathrm{cdga}^{\leq 0}[\mathrm{qiso}^{-1}]$ ∞-category.

a derived stack is a functor $F: d\mathrm{Aff}^{op} \to \mathrm{Spaces}$ satisfying the above gluing condition.

<u>Examples</u>: • $A \in cdga^{\leq 0}$ $\mathrm{Spec}(A) : B \to \mathrm{Hom}(A, B)$

• simplicial diagram of affine schemes $X_\bullet$

$$\underline{X_\bullet} : B \longmapsto |X_\bullet(B)|$$

(stackifies to a derived stack)

e.g. G affine gp scheme $\underline{N(G)}$

associated derived stack is BG.

• $G \circlearrowright X$, $[X/G]$ stack associated to $\left(\begin{matrix} \cdots \\ G^2 \times X \\ \downarrow\downarrow\downarrow \\ G \times X \\ \downarrow\downarrow \\ X \end{matrix} \right)$

• M space $\underline{M} : B \longmapsto M$ associated stack is M_B.

<u>Forms and closed forms on derived stacks</u>

$$DR : dAff^{op} \longrightarrow \varepsilon\text{-}Cpx^{gr} = \text{graded mixed cpx.}$$

graded complx = $\mathbb{Z}$-graded object in the category of complexes

2-gradings ⇝ degree means cohomological degree
⇝ weight means auxiliary grading.

graded mixed complx = graded complex C together with a morphism $\varepsilon : C \to C[1](1)$ such that $\varepsilon^2 = 0$.

i.e. two differentials one of deg $(\overset{c}{1}, \overset{w}{0})$ and the other one of degree $(1,1)$.

$DR(A) := S_{\tilde{A}}\left(\Omega^1_{\tilde{A}}[-1]\right)$ weight p part is the S^p part

$d_{int} = d_{\tilde{A}}$

$d_{dR} =$ has degree 1 and weight 1.

What is $\tilde{A}$?

$k \longrightarrow A$ replace by a cofibration.

$$\begin{tikzcd} & \tilde{A} \arrow[dr, "\wr"] & \\ k \arrow[ur] \arrow[rr] & & A \end{tikzcd}$$

$\tilde{A}$ is a semi free cdga → underlying graded algebra

Meaning that $\tilde{A}^{\#}$ is a polynomial algebra in variables of ≤ 0 degree.

Proposition: DR satisfies the glueing condition

One can define $DR(X) := \lim\limits_{\mathrm{Spec}\, A \to X} DR(A)$

$X = [M/G]$ $\quad M$ smooth alg. variety

G reductive gp. scheme acting on M.

$$DR(X) = \left(\Omega^{\cdot}(X) \otimes S(\mathfrak{g}^*[-2])\right)^G$$

$d_{int} = \sum_i \iota_{\vec{x}_i} \otimes \xi^i \qquad (x_i)$ basis $\mathfrak{g}$

ξ^i basis of $\mathfrak{g}^*$

$\varepsilon = d_{dR} \otimes id$

<u>Definition</u>

- a <u>2-form of degree k</u> on a derived stack X is a pure weight d_{int}-cocycle of cohomological degree $2+k$
- a <u>closed</u> 2-form of degree k on X is a 2-cocycle for the total differential, d_{tot}, of weight ≥ 2 and degree $2+k$.

Assume ω_0 is a 2-form of degree k on X.

$$d_{int}(\omega_0) = 0$$

$$d_{dR}(\omega_0) = d_{int}(\omega_1) \qquad \omega_1 \text{ weight } 3$$

same degree

$$d_{dR}(\omega_1) = d_{int}(\omega_2)$$

$$\vdots$$

Damien Calaque III

More linear algebra in the ∞-cat. context

$V, W \in Cpx_\infty(k\text{-mod})$

- Hom space from V to W constructed from Hom complex from V to W
- points in the Hom space from V to W
  $\Leftrightarrow$ 0-cocycles in the Hom complex
  $\Leftrightarrow$ cochain map from V to W
- given cochain maps $f, g : V \to W$ a path between these is the data of h such that
  $$f - g = h d_V + d_W h =: d(h)$$

In particular a loop at 0 is a self homotopy of the 0 map, i.e. $h : V \to W[-1]$ s.t $d(h) = 0$.

∞-categorical formulation

$$\Omega_0 \mathrm{Hom}(V, W) \cong 0 \underset{\mathrm{Hom}(V,W)}{\times} 0 \cong \mathrm{Hom}(V, 0) \underset{\mathrm{Hom}(V,W)}{\times} \mathrm{Hom}(V, 0)$$

$$\simeq \mathrm{Hom}(V, 0 \underset{W}{\times} 0) = \mathrm{Hom}(V, W[-1])$$

In particular $\mathrm{presymp}(V, n-1) = \mathrm{Hom}(\Lambda^2 V, k[n-1])$

$$\cong \Omega_0 \mathrm{Hom}(\Lambda^2 V, k[n]) \simeq \mathrm{Isotr}(V \to 0(n))$$

<u>Claim</u> The two notions of N.D. coincide

① $\omega \in \mathrm{Hom}(\Lambda^2 V, k[n-1])$ N.D. $\overset{\text{def.}}{\Longleftrightarrow}$ $V \to V^\vee[n-1]$ quasi-iso

② $0 \overset{\omega}{\sim} 0$ in $\mathrm{Hom}(\Lambda^2 V, k[n])$

$$\begin{tikzcd} V \arrow[r] \arrow[dr, dashed, "w"', "\text{q.iso}"'] & \mathcal{O} \cong \mathcal{O}[n] \arrow[r] & V^\vee[n] \\ & V^\vee[n-1] \arrow[u] \arrow[r] & 0 \arrow[u] \end{tikzcd}$$

$\Rightarrow \mathrm{Symp}(V, n-1) \cong \mathrm{Lagr}(V \to \mathcal{O}[n])$

__Rk.__ N manifold of dim n oriented compact.

$M = \partial N$

$\Omega^*(M,k)[1]$ is $(1-n)$-symplectic

$\Omega^*(N,k)[1] \to \Omega^*(M,k)[1]$ has Lagr. structure

If $M = \emptyset$ then $\Omega^*(N,k)[1] \to 0$.

__Def__ X derived stack

- the space $A^2(X,n)$ of 2-forms of deg. n is the space of weight 2 d_{int}-cocycles of deg $2+n$.
- the space $A^{2,cl}(X,n)$ of closed 2-forms of deg n is the space of weight ≥ 2 d_{tot}-cocycles of deg $2+n$.

If $w_0 + w_1 + \dots$ is a closed 2-form, w_0 is its underlying 2-form.

Rk. $\Omega_0 A^{2,cl}(X,n) \cong A^{2,cl}(X,n-1)$

Example

- $X = [M/G]$ M smooth alg. variety
  G reductive alg. gp. acting on M

a) $M = *$ $X = BG$ $DR(X) = S^{*}(\mathfrak{g}^{*}[-2])^{G}$

Let's pick an invariant sym 2-form on $\mathfrak{g}$.

(ξ^{i}) basis of $\mathfrak{g}^{*}$ orthogonal for pairing.

$$w = \sum_i \xi^i \xi^i \in S^2(\mathfrak{g}^*)^G = \left(S^2(\mathfrak{g}^*[-2])[4]\right)^G$$

$\Rightarrow$ w is a cocycle of weight 2 and deg 2+2

$\Rightarrow$ w is a closed 2-form of deg 2 on BG.

b) $G = \{1\}$ 2-forms on M are 2-forms of degree 0 on $X = M$.

c) $M = \mathfrak{g}^*$

(X_i) basis of $\mathfrak{g}$

(ξ^i) basis of $\mathfrak{g}^*$ dual to (X_i)

$$w = \sum_i d_{dR} X_i \xi^i \in (\Omega^1(M) \otimes \mathfrak{g}^*[-2])^G[3]$$

w is a closed 2-form of deg 1 on $[\mathfrak{g}^*/G]$.

d) $M = G$ and ~~[illegible]~~ G acts by conjugation.

$\langle , \rangle$ N.D. sym. inv. pairing on $\mathfrak{g}$.

$$w_0 = \frac{1}{2}(\theta_L + \theta_R) \in \left(\Omega^1(G) \otimes \mathfrak{g}[-2]\right)/[3]$$

$d_{int}(w_0) = 0$ $d_{dR}(w_0) = d_{int}(w_1)$

$$w_1 = \frac{1}{12} \langle \theta_K , [\theta_K, \theta_K] \rangle \in \Omega^3(G)^G$$

$w = w_0 + w_1$ is a closed 2-form of degree 1 on $[G/G]$

Assume that the derived stacks we consider are generic and l.f.p.

$$\exists\ \mathbb{L}_X \in QCoh(X) = \lim_{Spec(A) \to X} (A\text{-mod})$$

Prop. Weight 2 part of $DR(X)$ is equivalent to $\Gamma(X, S^2(\mathbb{L}_X[-1]))$.

If w is a closed 2 form of degree n then w_0 gives a morphism $\mathbb{T}_X = \mathbb{L}_X^\vee \longrightarrow \mathbb{L}_X[n]$.

Def. We say w is n-symplectic if w_0 is N.D. (i.e. $\mathbb{T}_X \to \mathbb{L}_X[n]$ is an equivalence)

Examples

- 2-symp. as soon as $\langle , \rangle$ is N.D.
- symp. structures on smooth M are 0-symp. structures on $X = M$
- $[\mathfrak{g}^*/G]$ is 2-symp.
- $[G/G]$ is 1-shifted symp. (G-reductive)

Let $f: L \to X$ be a morphism of derived stacks, generic and l.f.p. Let ω be an n-symp. on X.

<u>Def</u> A <u>Lagr. structure</u> on f is a path between $f^*\omega$ and 0 in $A^{2,cl}(L,n)$

$w = w_0 + w_1 + \dots$ What is a path γ between $f^*\omega$ and 0?

$$f^*\omega = f^*\omega_0 + f^*\omega_1 + \dots$$

$$f^*\omega_0 = d_{int}\gamma_0$$

$$f^*\omega_1 = d_{int}\gamma_1 + d_{dR}\gamma_1$$

$$\vdots$$

<u>Example</u> • $\mathrm{Lag}(X \to *(n)) = \mathrm{Symp}(X, n-1)$

• $M \xrightarrow{\mu} \mathfrak{g}^*$ G-equiv.

$\rightsquigarrow [\mu] = [M/G] \to [\mathfrak{g}^*/G]$

$[\mu]^* / \Sigma d_{dR} X_i \xi_i = \Sigma d_{dR} \mu^* X_i \xi^i = d_{int}(X)$

$\gamma \in \Omega^2(M)^G$ $d_{int}(\gamma) = \Sigma L_{\vec{X}}(\gamma)\xi^i$

$\Rightarrow \forall X \in \mathfrak{g}$ $L_{\vec{X}}(\gamma) = \mu^* d_{dR} X$

that condition is that $d_{dR}(\gamma) = 0$.

Thm [PTW] (Y,ω) n-shifted symp.

X is $\nearrow$ either M_B (M or. complex mfld dim d)
$\searrow$ smooth d-CY proj. variety

Then $\mathrm{Map}(X,Y)$ admits an $(n-d)$-shifted symp. structure.

$$\mathrm{Map}(X,Y)(A) := \mathrm{Hom}_{dst}(X \times \mathrm{Spec}\, A, Y).$$