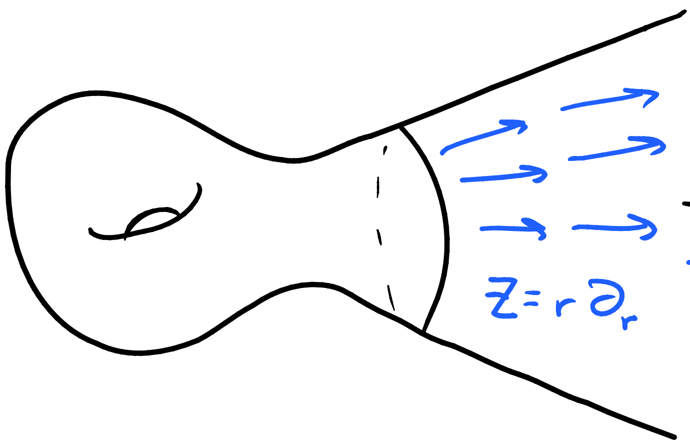
# Ganatra - Complementary Lecture 1

## §0. More about wrapped Fukaya categories

$(X^{2n}, \lambda)$ = Liouville mfld, presented as

$$X = \bar{X} \cup \partial\bar{X} \times [1,\infty)_r$$




- $d\lambda$ symplectic
- Convexity

$\underline{\underline{Z}}$ = Liouville v.field ($i_Z\omega = \lambda$) outward near $\infty$

**Examples:**

(a) Cotangent bundles

(b) Affine varieties

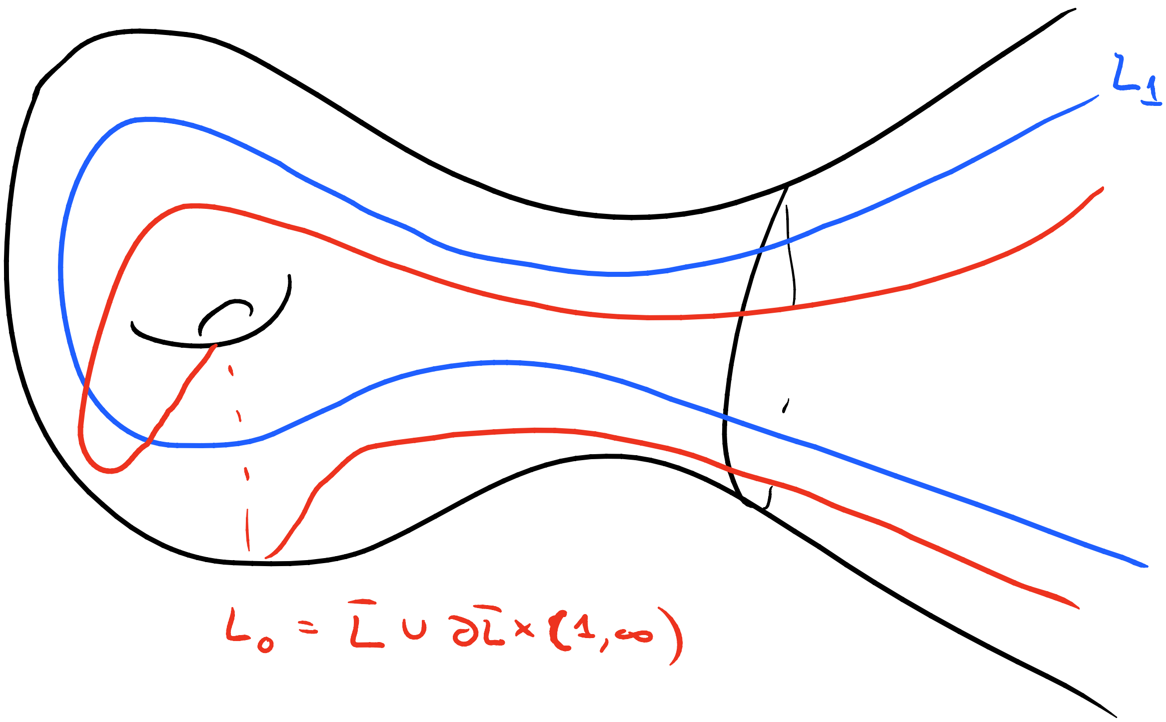
$\rightsquigarrow$ More generally, Stein/Weinstein mflds

$$(X, \lambda) \rightsquigarrow \mathcal{W}(X) = \text{wrapped Fukaya category}$$

<u>Objects</u>: Exact Lagns, conical near $\infty$

$f : L \to \mathbb{R}$

$df = \lambda|_{\mathbb{R}}$ $\quad f \equiv 0$ near $\infty$




$L_0 = \bar{L} \cup \partial\bar{L} \times [1,\infty)$

Roughly,

$$\hom_W(L_0, L_1) = \text{``}\, \Bbbk\langle L_0 \pitchfork L_1 \rangle \oplus \Bbbk\left\langle \begin{smallmatrix}\text{Reeb chords} \\ \partial\bar{L}_0 \to \partial\bar{L}_1\end{smallmatrix} \right\rangle \text{''}$$

$$= CW^*(L_0, L_1)$$

$$:= CF^*(L_0, L_1; H_{quadratic})$$

$H_q : X \to \mathbb{R}$

$= r^2$ near $\infty$

(or any $h(r)$ with $h'(r) \to \infty$)

$$\cong CF^*(\phi_{H_q}(L_0), L_1)$$

## General Features:

(a) Cotangent bundles: Earlier today (Auroux Part 1),

for $X = T^*S^1$,

$$L_0 = T_q^* S^1$$

then

$$CW^*(L_0, L_0) \simeq HW^*(L_0, L_0)$$
$$= \mathbb{k}[z, z^{-1}]$$
$$= \mathbb{k}[\pi_1(S^1)]$$

More generally, for $Q$ a compact mfld,

**Thm** [Abbondondalo-Schwarz on homology level; Abouzaid on chain level]:

$$CW^*(T_q^*Q, T_q^*Q) \underset{A_\infty}{\simeq} C_{-*}(\Omega_q Q)$$

(when $Q$ is spin, o/w need to twist on one side)

algebra from $\Omega_q Q \times \Omega_q Q \to \Omega_q Q$

$L\Big($ one side $\Big)$ $\qquad \Omega_q Q \to \Omega_q Q$

In general, $Q \rightsquigarrow P(Q)$ path category

$$\operatorname{ob} P(Q) = \text{points } p \in Q$$

$$\operatorname{Mor}_{P(Q)}(p,q) = C_*(P_{p,q}(Q))$$

Same result shows:

$$W^{\text{Fibers}}(T^*Q) \underset{A_\infty}{\simeq} P(Q)$$

Also,

<u>Thm</u> [Abouzaid]:

Any cotangent fiber generates $W(T^*Q)$

$$\Longrightarrow D^\pi W(T^*Q) \cong \underset{(\text{perf})}{\text{mod}}\text{-}P(Q)$$ $\rightsquigarrow$ derived local systems

$$\cong \operatorname{perf} C_* \Omega_q Q$$

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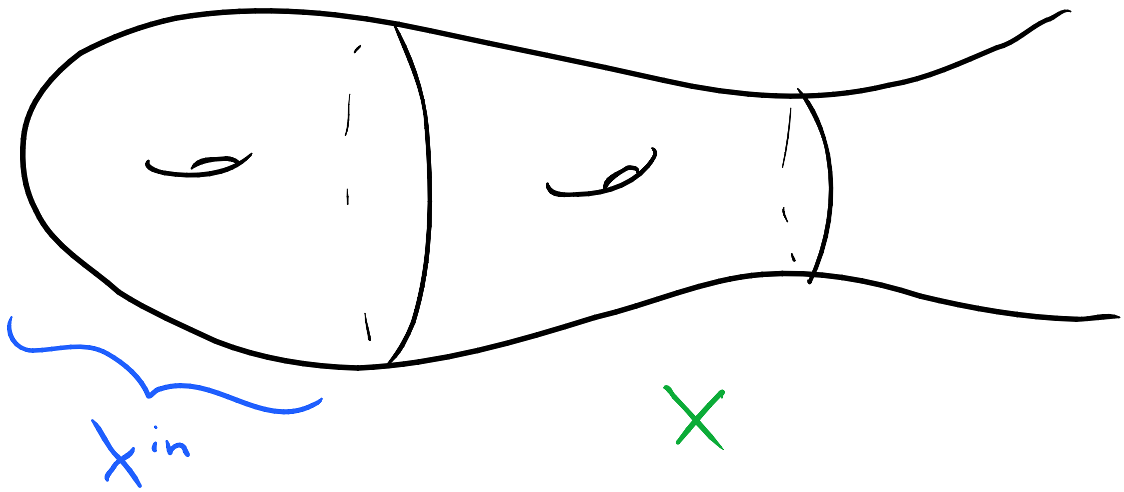
## (b) Viterbo-style restriction functor

[Abouzaid-Seidel] Under suitable hypotheses:




$$\begin{array}{ccc} \mathcal{W}(X) & \longrightarrow & \mathcal{W}(X^{in}) \\ L & \longmapsto & L \cap X^{in} \end{array}$$

(completed to something conical)

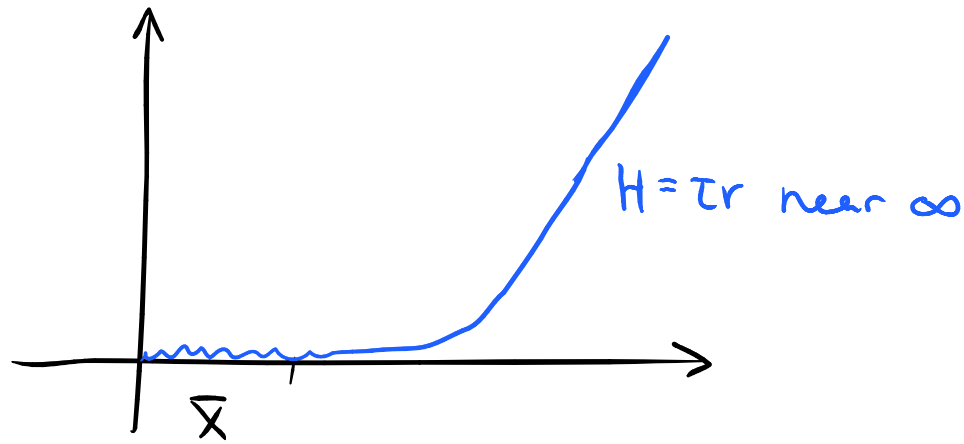
## (c) Direct limit formulation of wrapped Fukaya categories

Alternate defn (in homology)

linear Hamiltonian of slope $\tau$

Define $HW^*(L_0, L_1) = \varinjlim_{\tau} HF^*(L_0, L_1; H_\tau)$

$\xrightarrow[\tau]{} HF(L_0, L_1, H_\tau)$

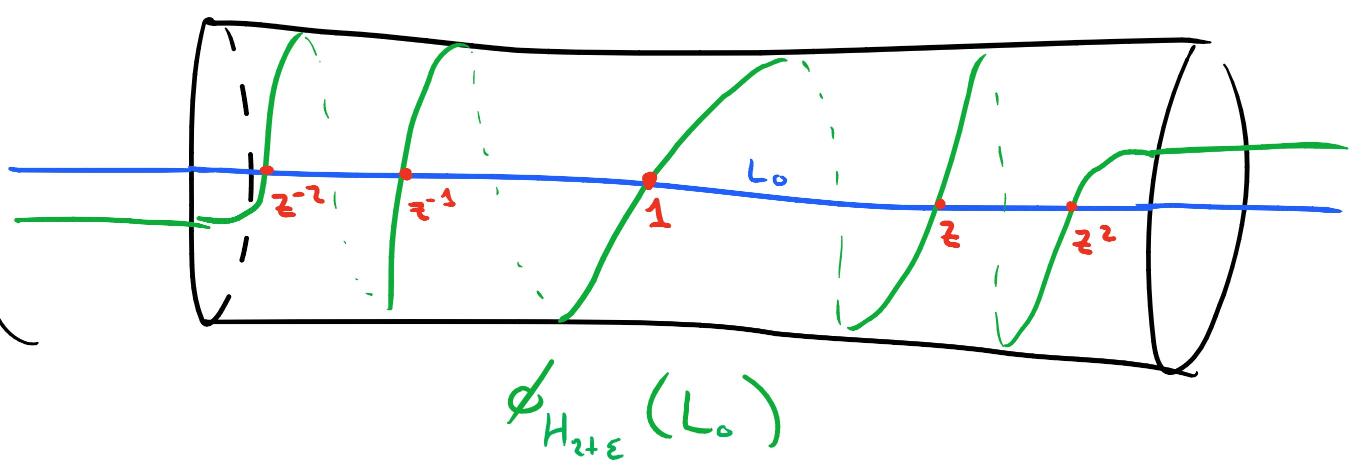



$\rightsquigarrow$ "homology gen'd by interior and Reeb chords of lengths $< \tau$"

($\tau$ generic, no Reeb chords of length $= \tau$)

Rmk: Chain-level operations require an interpretation of "homotopy direct limit"

Picture:



$\phi_{H_{2+\varepsilon}}(L_0)$

## §2. Wrapped Fukaya Categories as localizations

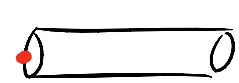
[Abouzaid-Seidel, Sylvan]

The $\underline{HF^*_\tau(L_0, L_1)}$ groups have interpretations in their own right as morphisms in a category.

$$= HF^*(\phi_{H_\tau}(L_0), L_1)$$

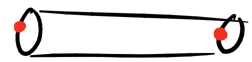
We'll focus specifically on this family of examples.

| B-side | | A-side |
|---|---|---|
| $\mathbb{C}^*$ | $\longleftrightarrow$ | $\mathbb{C}^*$ |
| $\cap$ | | |
| $\mathbb{C}$ | $\longleftrightarrow$ | $(\mathbb{C}^*, W = \frac{1}{z}) \rightsquigarrow \mathbb{C}^*$, some data near $\infty$ |
| $\cap$ | | |
| $\vdots$ | | |



$\mathbb{CP}^1 \longleftrightarrow \left( \mathbb{C}^*, W = z + \frac{1}{z} , \right)$

$\mathbb{C}^*$, some data near 0 and $\infty$



## Two different types of Fukaya categories of non-compact Lagrangians

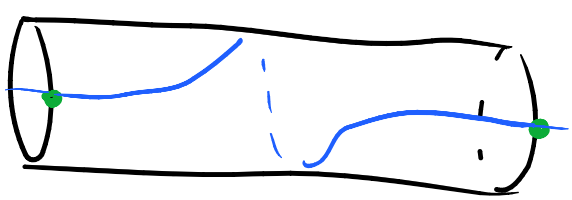
(there is also directly $\mathcal{F}(X^\vee, W) = FS(X^\vee, W)$)

### ⓐ Infinitesimal Fukaya Category [Nadler-Zaslow]

Data: $X$ Liouville

$A \subset \partial X$ subset of "asymptotics" (Legn, ...)

Objects: $L$ w/ $\partial L \subset A$



Homs

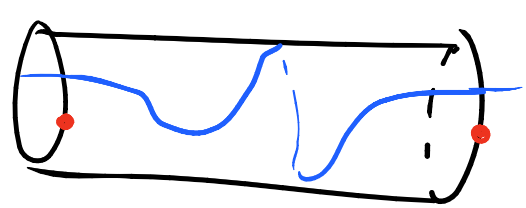
### ⓑ Partially-Wrapped Fukaya Category [Abouzaid-Seidel, Sylvan, Abouzaid-Auroux]

Data: $X$ Liouville

$B \subset \partial X$ (Legn/sympl, ...)

"barriers" (stops [Sylvan])

Objects: $L$ w/ $\partial L \cap B = \emptyset$



Homs:

$$\hom_{\mathcal{F}_{inf}}(L_0, L_1)$$
$$= CF^*(\phi^{\varepsilon} L_0, L_1)$$

time 1 flow of linear Ham of slope $\varepsilon$

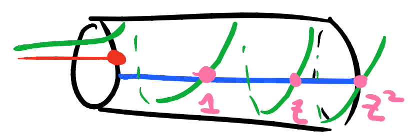
$$H^* \hom_{W_p}(L_0, L_1)$$
$$= \varinjlim HF^*(\phi_H(L_0), L_1)$$

$H$ = large "positive Hamiltonian" which is $\equiv 0$ in small nbhd of $B$

# Example:

| B-side | A-side |
|---|---|
| $\mathbb{C}$ | $(\mathbb{C}^*, \frac{1}{z})$ |
| $\mathcal{O} \in \mathrm{Ob}\, \mathrm{Coh}(\mathbb{C})$ | $T^*_{z_0} S^1$ |

$Ext(\mathcal{O}, \mathcal{O})$ = fns on $\mathbb{C}$
$= \mathbb{k}[z]$




Get all positive powers of $z$ from partial wrapping