Ganatra – Part 2

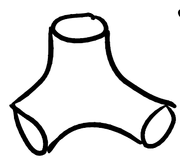
§1. $\mathcal{FS}(X^\vee, W)$ $\xleftrightarrow{\text{yesterday}}$ $\mathcal{W}_{\text{partially}}(X^\vee, B = W^{-1}(\infty))$

+ its structures (intrinsically, w/o)

$\wr\|$ sometimes

$\left(\mathcal{F}_{\text{inf}}(X^\vee, A = W^{-1}(+\infty))\right)$

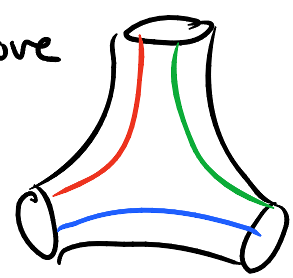
§2. $\mathcal{W}(X)$ as a localization

§3. $\mathcal{W}\Big($  $\Big)$ via Lag'n skeleton

§4. Generation criteria after Abouzaid

[+Abouzaid-FOOO, Abouzaid-G]

To see how to prove



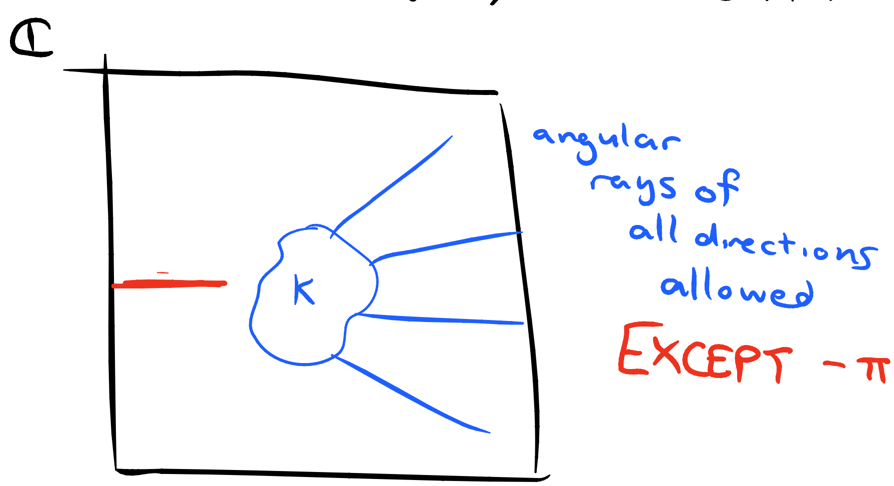
split-generates $\mathcal{W}(X)$

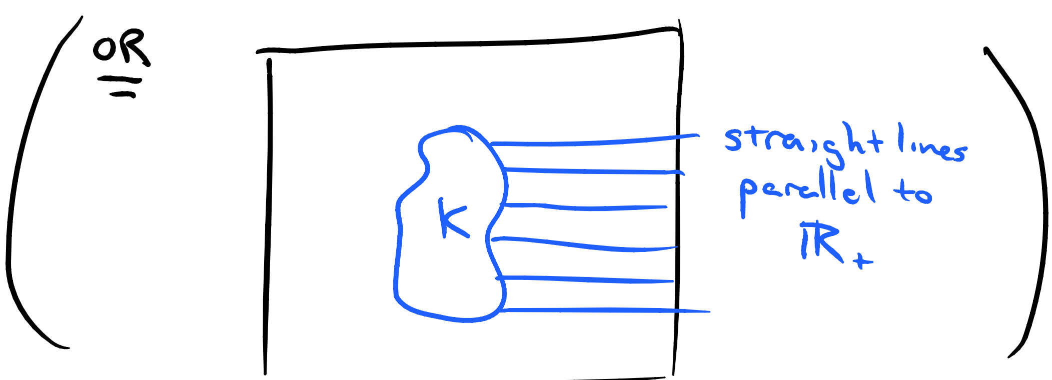
§1. $\mathcal{FS}(X^\vee, W)$ [Kontsevich, Seidel (w/ Lefschetz), Abouzaid–Seidel]

$$W : X^\vee \longrightarrow \mathbb{C}$$

Objects: $L \subset X^\vee$

s.t. $W(L)$ contained in

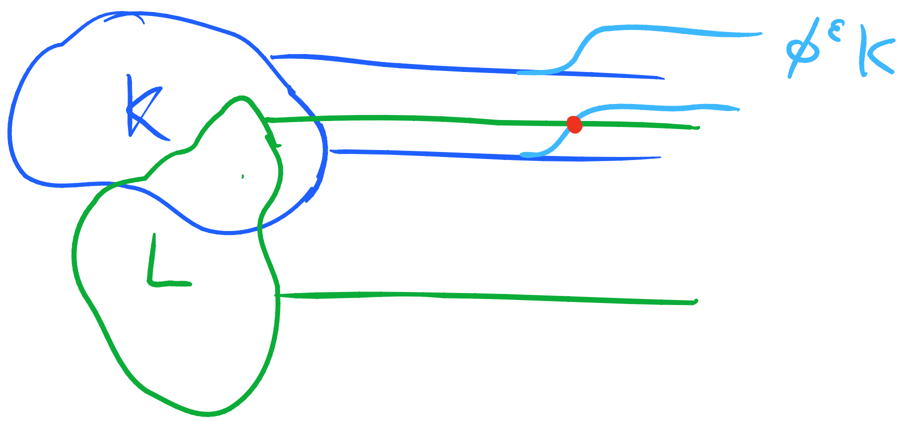




Homs:

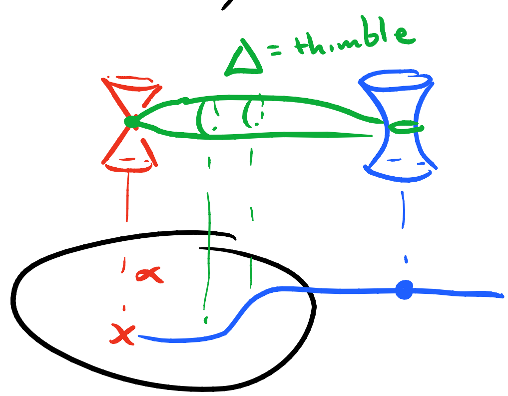
$$\hom_{FS(X^\vee, W)}(K, L) = CF^*(\underset{=}{\phi^\varepsilon} K, L)$$

time ε bend in base, ε large enough so all ends of K are $>$ all ends of L



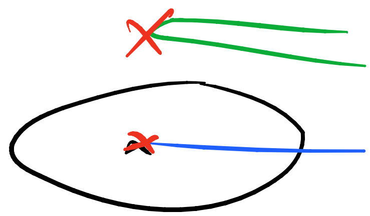
Sample objects

- When W = Lefschetz fibn, get thimbles



- When W = Lefschetz-Morse-Bott,

L -



Δ_L generalized thimble

$(\mathbb{R}^*\text{-bdle over } L \subseteq M_{crit})$

Mirror symmetry table: X Fano, $D \subset X$ anticanonical

	B-side		A-side	
$\mathbb{C}P^1$	$\mathrm{Coh}(X)$	$\longleftrightarrow$	$\mathcal{FS}(X^\vee, W)$	$(\mathbb{C}^*, z+\frac{1}{z})$
$\bullet\ \bullet$	$\mathrm{Coh}(D)$	$\longleftrightarrow$	$\mathcal{F}(W^{-1}(\infty))$	$\bullet\ \bullet$
$\mathbb{P}^1 \setminus \{0,\infty\} \cong \mathbb{C}^*$	$\mathrm{Coh}(X \setminus D)$	$\longleftrightarrow$	$\mathcal{W}(X^\vee)$	$\mathbb{C}^*$

(Example of $\mathbb{C}P^1$ above)

Expectation:

HMS should be compatible w/ functoriality on both sides

$$D \overset{i}{\subset} X \overset{j}{\supset} X \setminus D$$

induces, e.g.

$$j^* : Coh(X) \to Coh(X \setminus D) \longleftrightarrow \mathcal{FS}(X^\vee, w) \xrightarrow{\text{accel.}} \mathcal{W}(X^\vee)$$

$$\left(HF^*(\phi^\varepsilon K, L) \to \varinjlim_\tau HF^*(\phi^\tau K, L) \right)$$

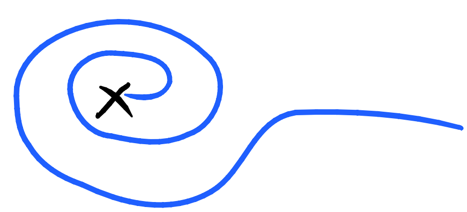
$$i^* : Coh(X) \to Coh(D) \longleftrightarrow \mathcal{FS}(X^\vee, w) \xrightarrow{\cap} \mathcal{F}(w^{-1}(\infty))$$

$$\left(L \longmapsto \text{“}\partial^\infty L\text{”} \subset w^{-1}(\infty) \right)$$

Observe:

In $\mathcal{W}(X)$, L and $\phi_{H_\tau}(L)$ are isomorphic objects

vs.



follows immediately from

)

$$\mathrm{Hom}_w(\phi_{H_\tau}L, K) = \varinjlim_{s} HF^*(\phi_{H_s}\phi_{H_\tau}L, K)$$

$$\mathrm{Hom}_w(L, K) = \varinjlim_{s} HF^*(\phi_{H_s}L, K)$$

(cofinality)

and moreover, $\exists$ a natural morphism in $\mathcal{FS}(X^\vee, w)$

$\phi_{H_\tau}L \to L$ inducing isomorphism

continuation element

Conclusion:

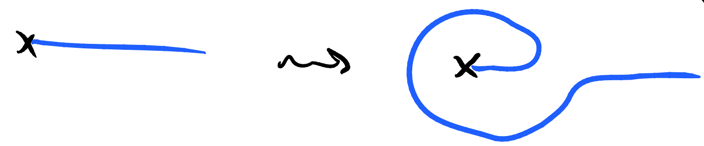
$$\begin{tikzcd}
\mathcal{FS}(X^\vee, w) \arrow[rr, "\mathrm{accel}"] \arrow[dr, dashed] & & \mathcal{W}(X^\vee) \\
& \mathcal{FS}(X^\vee, w)[z^{-1}] \arrow[ur, "\cong"'] &
\end{tikzcd}$$

[Seidel, Abouzaid-Seidel]

morphisms $\phi_{H_\tau}L \to L$

Note: In $\mathcal{FS}(X^\vee, W)$, suffices to study

ϕ_{H^τ} for $\tau \in \mathbb{Z}_{\geq 0}$

(e.g. ϕ_{H^1} : )

<u>Propn</u> [Kontsevich, Seidel]:

ϕ_H is mirror to the (inverse) Serre functor

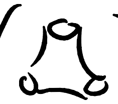
$$\mathcal{FS}(X^\vee, W)[z^{-1}] = \mathcal{FS}(X^\vee, W) \Big/ \begin{array}{l} \text{geom.} \\ \text{nat trans} \\ \text{id} \to \text{Serre}^{-1} \\ \text{or Serre} \to \text{id} \end{array}$$

$\updownarrow$

element

$$\xi \in HF^*(\phi_H L, L)$$

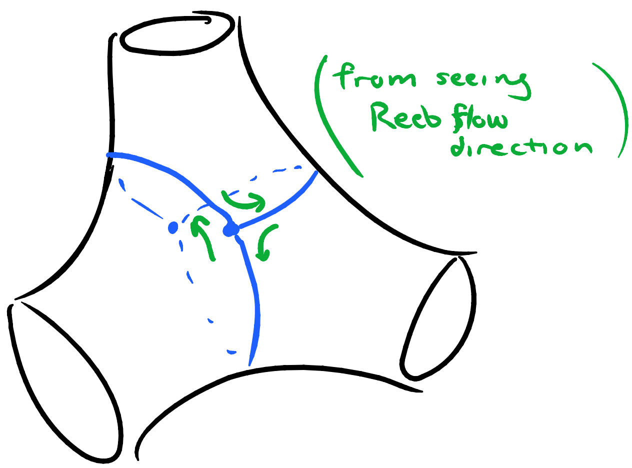
continuation element

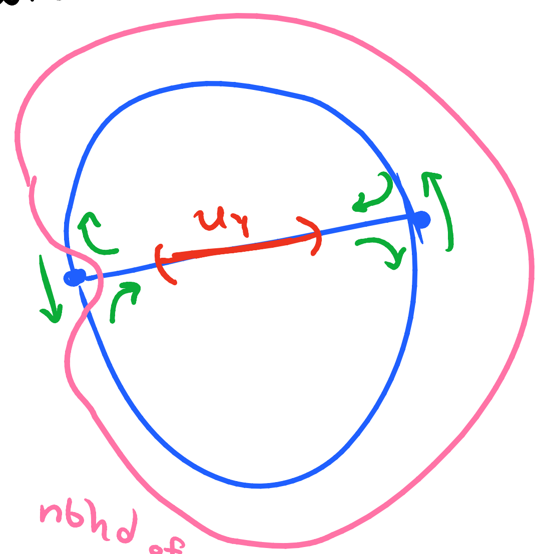
§3. W() from skeleton

[toy case, G - Pardon - Shende]

Many choices of skeleta (known for a long time)

Maximally symmetric choice:





Cosheaf of categories

$$\mathcal{C} : \text{Open sets} \longrightarrow \text{Cat}$$

with co-restrictions

$$\begin{array}{ccc}
\mathcal{C}(\mathcal{U}_X) & & \mathcal{C}(\text{left tripod}) \\
\mathcal{C}(\mathcal{U}_Y) & & \\
\mathcal{C}(\mathcal{U}_Z) & & \mathcal{C}(\text{right tripod})
\end{array}$$

What are they?

path category

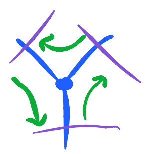
$$\mathcal{C}(\mathcal{U}_x) = \mathcal{P}(\mathcal{U}_x) \simeq \mathbb{K}$$

write as:

• L_x

one object L_x

$\mathrm{End}(L_x) = \mathbb{K}$

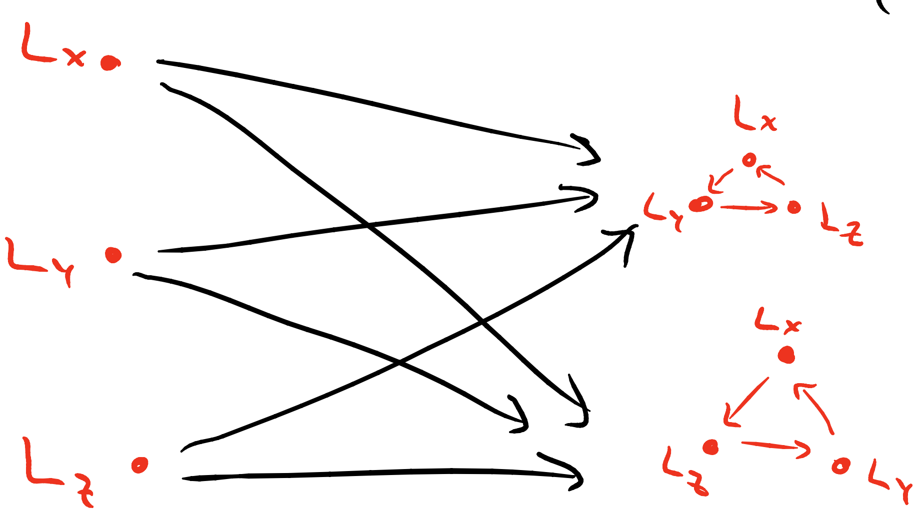
$\mathcal{C}\Bigg($  $\Bigg) =$

$$\begin{tikzcd} & L_x \arrow[dl, "z"'] & \\ L_y \arrow[rr, "x"'] & & L_z \arrow[ul, "y\ {[1]}"'] \end{tikzcd}$$

Successive compositions $= 0$

$+\ \mu^3(z,y,x) = \mathrm{id}$

(+ cyclic)

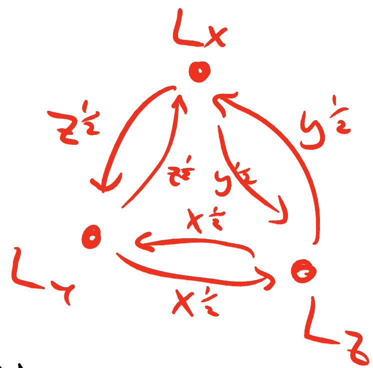


Global sections of $\mathcal{C}$

$= \operatorname{hocolim}(\text{diagram})$

Exercise This is described by

paths on the quiver:



s.t. any path going through all three vertices vanishes!

$+ \mu^3$ relations