MICROLOCAL THEORY OF SHEAVES AND SYMPLECTIC TOPOLOGY I

Guiraux, I
27/06/16

- $\mathcal{C}$ category, $\mathcal{C}^{op}$ opposite category, $\mathcal{C}^\wedge = \text{Functors}(\mathcal{C}^{op}, \text{Set})$ Guiraux

Yoneda functor $c : \mathcal{C} \hookrightarrow \mathcal{C}^\wedge$
$x \longmapsto x^\wedge = \text{Hom}_\mathcal{C}(*, x)$

<u>Yoneda's lemma</u> : c is fully faithful

$\leadsto$ image of c = representable functors

- <u>Limits</u>

Fix a category $\mathcal{I}$ (set of indices)

<u>Def</u> An <u>inductive system</u> in $\mathcal{C}$ indexed by $\mathcal{I}$ is a functor $F : \mathcal{I} \to \mathcal{C}$. A <u>projective system</u> in $\mathcal{C}$ indexed by $\mathcal{I}^{op}$ is a functor $F : \mathcal{I}^{op} \to \mathcal{C}$

To define limits, we proceed as follows:

<u>Step 1</u> : proj. limits for $\mathcal{C} = \text{Sets}$

Let $F : \mathcal{I}^{op} \to \text{Set}$ be a proj. system

Then $\varprojlim_{\mathcal{I}} F$ is the set $\left\{ (x(i))_{i \in ob(\mathcal{I})} \;\middle|\; \begin{array}{l} \forall f : i \to j \\ F(f)\, x(j) = x(i) \end{array} \right\}$

Rmk Here $x(i) \in F(i)$
So this makes sense

$\forall x \in ob(\mathcal{C})$

<u>Step 2</u> inductive and projective limits

$F : \mathcal{I} \to \mathcal{C}$ ind. system, $i \longmapsto \{\text{Hom}_\mathcal{C}(F(i), x)\}$ proj. syst in Set

<u>Def</u> $\varinjlim_{\mathcal{I}} F(x) = \varprojlim_{i \in \mathcal{I}} \text{Hom}_\mathcal{C}(F(i), x)$

If $\varinjlim_{\mathcal{I}} F$ is representable, we say that F admits an inductive limit in $\mathcal{C}$, we still denote it by $\varinjlim_{\mathcal{I}} F$

By def of representability, we get that $\varprojlim_{\mathcal{I}} \text{Hom}_\mathcal{C}(F(i), x) = \text{Hom}_\mathcal{C}(\varinjlim_{\mathcal{I}} F, x)$

Same construction for projective systems:

$F: \mathcal{J}^{op} \to \mathcal{C} \rightsquigarrow \{\mathrm{Hom}_{\mathcal{C}}(x, F(i))\}_{i \in \mathcal{J}}, \forall x \in ob(\mathcal{C})$

$\rightsquigarrow \varprojlim F \in ob(\mathcal{C}^{\wedge}) \rightsquigarrow \varprojlim F(x) = \varprojlim \mathrm{Hom}_{\mathcal{C}}(x, F(i))$

$[\mathcal{C}^{\wedge} = Fct(\mathcal{C}^{op}, Set)]$

<u>Step 3</u> inductive lim in Sets

Assume $\mathcal{J}$ is <u>filtrant</u>, ie

$\forall i, j \in ob(\mathcal{J})\ \exists k \quad i \to k,\ j \to k$

$\forall i, j, \forall f, g \in \mathrm{Hom}(i, j)\ \exists (k, h)$ s.t. $i \overset{f}{\underset{g}{\rightrightarrows}} j \xrightarrow{h} k$

$h \circ f = h \circ g$

If $\mathcal{J}$ is a filtrant category, inductive limits in Set parametrized by $\mathcal{J}$ are representable by $\varinjlim_{\mathcal{J}} F = \coprod_{i \in ob(\mathcal{J})} F(i) \Big/ \sim$, where

$x^{\in F(i)} \sim y^{\in F(j)} \Leftrightarrow \exists k,\ i \xrightarrow{f} k,\ j \xrightarrow{g} k,\ F(f)x = F(g) \cdot y$

<u>Step 4</u> Formal limits (project.) and colimits (induct)

Take $\mathcal{J}$ filtrant, consider $\mathcal{J} \xrightarrow{F} \mathcal{C} \xhookrightarrow{\text{Yoneda}} \mathcal{C}^{\wedge}$.

As a functor $\mathcal{J} \to \mathcal{C}^{\wedge}$, its inductive limit is representable

$\varinjlim_{\mathcal{J}} \mathrm{Hom}_{\mathcal{C}}(x, F(i)) \in ob(\mathcal{C}^{\wedge})$, it is $\varinjlim (\iota \circ F)$;

when the ind. limit of F does not exist, this is the correct notion to use and we denote it "$\varinjlim_{\mathcal{J}}$" F.

<u>Def</u> An object in $\mathcal{C}^{\wedge}$ iso. to "$\varinjlim_{\mathcal{J}}$" F is called an <u>ind-object</u>

We can similarly define <u>projective objects</u>

II| Additive & abelian category

(a) Fix a base ring k. A category is k-linear if all the Hom sets are k-modules (+ other condit., but let's forget about these)

~~If $k=\mathbb{Z}$, additive category~~

If $k=\mathbb{Z}$, we talk about additive category (instead of $\mathbb{Z}$-linear)

Take $\mathcal{C}$ additive category ($=\mathbb{Z}$-linear); consider then

$C^b(\mathcal{C}) = \{$ bounded complexes in $\mathcal{C} \}$ $\cdots \to 0 \to X_n \xrightarrow{f_n} \cdots \to X_m \xrightarrow{f_m} 0 \to \cdots$

Category with objects: $\cdots \to 0 \to X_n \xrightarrow{f_n} \cdots \to X_m \to 0 \to \cdots$, $f_i \circ f_{i-1} = 0$

Morphisms

$$\begin{array}{ccc} X_n & \longrightarrow & X_{n+1} \\ {\scriptstyle \varphi_n}\downarrow & \boxed{} & \downarrow{\scriptstyle \varphi_{n+1}} \\ X'_n & \longrightarrow & X'_{n+1} \end{array}$$

Then this category $C^b(\mathcal{C})$ admits a $C^b(k)$-enrichment,

ie $\underline{\mathrm{Hom}}(X_\cdot, Y_\cdot) \in C(k)$, $\underline{\mathrm{Hom}}(X_\cdot, Y_\cdot)_n \overset{\text{def}}{=} \bigoplus_{p\in\mathbb{Z}} \mathrm{Hom}(X_p, Y_{n+p})$

Rmk We talk about bounded complexes, but there are analogues (more complicated) for upper/lower bounded $C^{\pm}(\mathcal{C})$

with $\delta : \underline{\mathrm{Hom}}(--)_n \longrightarrow \underline{\mathrm{Hom}}(--)_{n+1}$

$$\delta\varphi = f^Y_{n+p} \circ \varphi + (-1)^{p+1} \varphi \circ f^X_n$$

We remark $\mathrm{Hom}_{C^b(\mathcal{C})}(X_\cdot, Y_\cdot) = Z^0(\underline{\mathrm{Hom}}(X_\cdot, Y_\cdot))$

closed w.r.t. δ

Def The homotopy category of $\mathcal{C}$, $K^b(\mathcal{C})$, has the same objects as $C^b(\mathcal{C})$ but $\mathrm{Hom}_{K^b(\mathcal{C})}(X_\cdot, Y_\cdot) \overset{\text{def}}{=} H^0(\underline{\mathrm{Hom}}(X_\cdot, Y_\cdot))$

Concretely: identify μ and ν if

$$\left[\ \mu - \nu = \delta_{-1}\varphi = f^Y \circ f + \varphi \circ f^X\ \right]$$

(b) Abelian categories:

Gavrilovic

Take k-lin. category $\mathcal{C}$, $f: x \to y$ morphism in $\mathcal{C}$

Def A kernel of f is the proj limit of $x \underset{0}{\overset{f}{\rightrightarrows}} y$ — 0 → zero morphism in $\mathrm{Hom}_{\mathcal{C}}(x,y)$

A Cokernel of f is the ind. limit of $x \underset{0}{\overset{f}{\rightrightarrows}} y$

eg
$$\begin{tikzcd} z \arrow[r] \arrow[rr, bend left, "0"] \arrow[dr, dashed, "\exists"'] & x \arrow[r, "f"] & y \\ & \mathrm{Ker} f \arrow[u] & \end{tikzcd}$$

Def A category is abelian if: • it admits Ker, coker
• $\forall f$ morph, $\mathrm{Im} f \to \mathrm{Coim} f$ is an iso

Here $\mathrm{Im} f \overset{\text{def}}{=} \mathrm{Ker}(y \to \mathrm{coker} f)$
$\mathrm{Coim} f \overset{\text{def}}{=} \mathrm{coker}(\mathrm{Ker} f \to x)$

If $\mathcal{C}$ abelian cat.,
$$\begin{tikzcd} C^b(\mathcal{C}) \arrow[r, "H^p"] \arrow[d] & \mathcal{C} \\ K^b(\mathcal{C}) \arrow[ur, dashed, "\exists \text{ factorization}"'] & \end{tikzcd}$$
$(x_n, f_n) \longmapsto \mathrm{Ker} f_p / \mathrm{Im} f_{p-1}$

Fact $K^b(\mathcal{C})$ is <u>not</u> abelian (no kernels, cokernels)
But still it has some useful structure, namely it is a triangulated categ. (see later)

Cone: Take $u: x_\cdot \to y_\cdot$, define $\mathrm{Cone}(u)_n = x_{n+1} \oplus y_n$
with differential given by $\begin{pmatrix} (-1)^n f^x_{n+1} & 0 \\ u_{n+1} & f^y_n \end{pmatrix}$

$\Rightarrow$ Exact sequence $0 \to y_\cdot \to \mathrm{Cone}(u) \to x_\cdot[1] \to 0$ (shifted by 1)

More in general, we call distinguished triangle an exact seq $x_\cdot \to y_\cdot \to \mathrm{Cone}\,u \to x(1) \to$

(c) Triangulated categories

Def $\mathcal{C}$ k-linear, $T: \mathcal{C} \to \mathcal{C}$ + collection of triangles $x \to y \to z \to Tx$
+ other axioms is called a <u>triangulated cat.</u>

Cones in a triang. cat.:

given $x \xrightarrow{u} y$, $\exists$ a dist. triangle $x \xrightarrow{u} y \to z \to Tx$

Moreover, • $\begin{array}{ccccccc} x & \to & y & \to & z & \to & Tx \\ \downarrow & & \downarrow & & \downarrow\exists & & \downarrow \\ x' & \to & y' & \to & z' & \to & Tx' \end{array}$ ⚠ This is NOT unique

• $x \to y \to z \to Tx$ D.T. (dist. triangle) $\Rightarrow$ $y \to z \to Tx \to Ty$ is D.T.

• octahedron axiom

d) Derived category

Actually, we can factorize more!
$$\begin{array}{ccc} C^b(\mathcal{C}) & \xrightarrow{H^0} & \mathcal{C} \\ \downarrow & \nearrow & \uparrow\exists \\ K^b(\mathcal{C}) & \longrightarrow & D^b(\mathcal{C}) \end{array}$$

Def A morphism u in $\begin{cases} C^b(\mathcal{C}) \\ K^b(\mathcal{C}) \end{cases}$ is a <u>quasi-iso</u>

if $\forall p$, $H^p(u)$ is an isomorph.

Def $D^b(\mathcal{C})$ has the same objects of $C^b(\mathcal{C})$ but all quasi-iso are formally inverted

ie if $\begin{array}{ccc} x_\bullet' & & \\ \downarrow \text{q-iso} & \searrow & \\ x_\bullet & & y_\bullet \end{array}$ in $C^b(\mathcal{C})$, $K^b(\mathcal{C})$, then $\begin{array}{ccc} x_\bullet' & & \\ \exists \uparrow\downarrow \text{q-iso} & \searrow & \\ x_\bullet & \xrightarrow{\exists} & y_\bullet \end{array}$ in $D^b(\mathcal{C})$

Moreover, $D^b(\mathcal{C})$ is still a triang. categ

and D.T. in $D^b(\mathcal{C})$ = ~~D.T~~ images of D.T. in $C^b(\mathcal{C})$

III Derived functors

$\mathcal{C}, \mathcal{C}'$ abelian categories.

$F: \mathcal{C} \to \mathcal{C}'$ left exact functor, ie $0 \to x \to y \to z \to 0$ exact $\Rightarrow$ $0 \to F(x) \to F(y) \to F(z)$ exact

<u>injective object in $\mathcal{C}$</u>: x is injective if $\mathrm{Hom}_{\mathcal{C}}(x, \cdot)$ is exact

Assume $\mathcal{C}$ has enough injective objects (ie every object is a subobject of an injective object)

Grivaux

The right derived functor of F, $RF : D^b(\mathcal{C}) \to D^+(\mathcal{C}')$ → we possibly unbounded at $+\infty$ but bounded at $-\infty$

is defined as follows:

- Take $X \in ob(D^b(\mathcal{C}))$. Then take a complex of injective ~~sheaves~~ objects quasi-isomorphic to X (see (i) of remark below)
- Apply F to each ~~sheaf~~ object in this complex of injective ~~sheaves~~ objects; this is our $RF(X)$ in $D^+(\mathcal{C}')$

RF is a triangulated functor

Define $R^pF := H^p \circ RF$

Moreover, to every ~~...~~ D.T. there's an associated long exact sequence with the images of the complexes by F

Rmk The definition of RF above works because:

(i) $K^+(\text{injective obj}) \xrightarrow[\sim]{\text{eq-cat}} D^+(\mathcal{C})$ and on the left cat RF is just F applied to each object of the inj. complexes

(ii) The definition we gave is "enough functorial"

Grivaux, lecture II

29/06/16

- Recommended lecture notes (for first 2 lectures):

"Microlocal theory of sheaves and symplectic top." P. Schapira's webpage

- For X top. space, we have $Psh(X) \underset{F \mapsto F^a}{\overset{\text{forgetful}}{\rightleftarrows}} Sh(X)$

presheaves ; sheafification

Rmk $Sh(X)$ can be understood as a localization:

localize (invert) all morphisms of pre-sheaves φ such that $\forall x \in X \quad \varphi_x : F_x \xrightarrow{\sim} G_x$

- Operations on sheaves

Take $f : X \to Y$. We can define:

(i) $f_* : sh(X) \to sh(Y)$
(ii) $f^{-1} : sh(Y) \to sh(X)$ } in adjunction

(iii) $f_! : sh(X) \to sh(Y)$
(iv) $f^! : D^+(sh(Y)) \to D^+(sh(X))$ } in adjunction (in derived category)

(V) Cutting a sheaf by a locally closed set

$j : Z \hookrightarrow X$ loc. closed, $F_Z := j_* j^! F$

If Z closed, $U := X \setminus Z$, Then $\underbrace{k_U \to k_X \to k_Z}_{\text{locally const. sheaves}} \xrightarrow{+1}$

(Vi) F sheaf on X, then we have $R\mathcal{H}om(\cdot, F)$

(Vii) For $F \in Sh(X)$, we define $\Gamma_Z F(U) := \{ s \in F(U) \mid \operatorname{supp}(s) \subset Z \}$

⚠ This is NOT the same as F_Z!

For example, we have $\Gamma_Z F = \mathcal{H}om(k_Z, F)$ (so $\Gamma_Z F$ is "often" zero, for ex. $\Gamma_Z k_X = 0$), while $F_Z = F \otimes_k k_Z$

We have $R\Gamma_Z F \to F \to i_* F|_U \xrightarrow{+1}$ ...

• ~~Microsupport of sheaves~~

• Small digression: stalks of sheaves

For $F \in Sh(X)$, we define The germ of F at x simply as $F_x := \varinjlim_{\substack{U \ni x \\ U \text{ open}}} F(U)$.

But This def does not extend to $D^b(X)$, because $\varinjlim$ does not extend there!

We remark that $F_x = i_x^{-1} F$, where $i_x : \{x\} \hookrightarrow X$; This extends without problem to $D^b(X)$, so we have a definition of stalk for each $F \in D^b(X)$ too.

• Microsupport of sheaves (Kashiwara - Schapira)

Fix X smooth mfld, $F \in D^b(X)$, $\pi : T^*X \to X$.

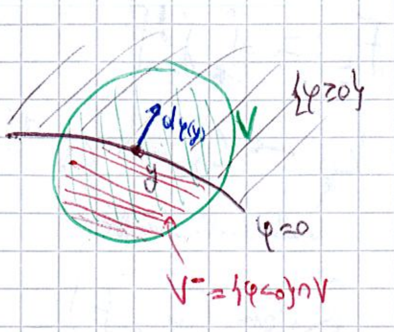
Def $SS(F)$ = "microsupport of F" is the subset of T^*X defined as follows:

$h \in T^*X$ is not in $SS(F)$ if $\exists$ nbhd U of h in T^*X s.t. $\forall y \in \pi(U)$, $\forall \varphi$ smooth funct. $\varphi : X \to \mathbb{R}$ with $\varphi(y) = 0$, $d\varphi(y) \in U$ we have $(R\Gamma_{\{\varphi \geq 0\}} F)_y = 0$

Gvorenx

Rmk By definition, $SS(F)$ is a closed subset of T^*X.

Picture:



Call $j : \{\varphi < 0\} \to 0$. Then $R\Gamma_{\{\varphi \geq 0\}} F \to F \to j_* F_{|\{\varphi<0\}} \xrightarrow{+1}$

which tells us that:

$$\left(R\Gamma_{\{\varphi\geq 0\}} F\right)_y = 0 \iff R\Gamma(V, F) \xrightarrow{\sim} R\Gamma(V^-, F)$$

if V small enough

<u>Morale</u> Cohomology classes (with coeff in F) extend locally in the directions that are outside $SS(F)$

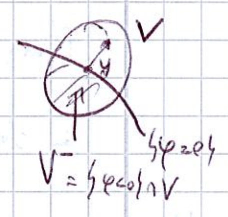
<u>Examples</u>:

(1) F locally constant sheaf on X

Then $SS(F) =$ zero section of T^*X

In fact, if $h_x \overset{\in T^*X}{}$ outside the zero section,

we easily get $R\Gamma(V^-, \overset{F}{k_X}) \xrightarrow{\sim} R\Gamma(V, k_X)$



so $(T^*X \setminus \underset{\to \text{zero set}}{0_X}) \cap SS(k_X) = \emptyset$.

The other inclusion follows from:

<u>Lemme</u> $\forall F$ sheaf (derived!), $SS(F) \cap \underset{\text{zero set of } T^*X}{0_X} = \operatorname{supp}(F)$

(2) $Z \subset X$ smooth closed submfld.

Then $SS(k_Z) = T^*_Z X$ (the conormal bundle to Z)

$\Rightarrow$ [propagation because $V^- \cap Z \neq \emptyset$ $\forall \varphi$, $\varphi(y) = 0$ if $d\varphi(y) \notin T^*_Z X$

In fact, we can see like before that for any other direct outside we have propagation and for those direct. ~~else~~ we ~~have~~ don't.

Giovanni

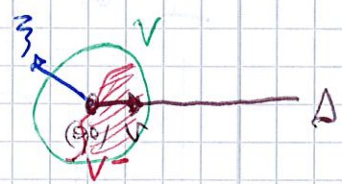
(3) $\Delta = [0,+\infty) \times \{0\}$ in $\mathbb{R}^2$, $F = k_{\mathbb{R}^2 \Delta}$

↳ it's the notation G_Z of before, with $G = k_{\mathbb{R}^2}$, $Z = \Delta$, so it gets a bit ugly...

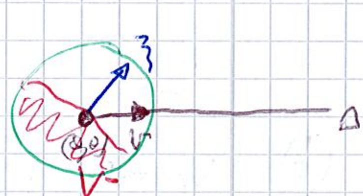
Locally near $y \in \Delta \setminus \{(0,0)\}$ it's like in example 2, so we understand it, it's $T^*_{\Delta \setminus \{(0,0)\}}\mathbb{R}^2$.

For $(0,0) \in \Delta$? We want those $\xi \in T^*_{(0,0)}\mathbb{R}^2$ s.t. $((0,0),\xi) \in SS(k_{\mathbb{R}^2\Delta})$

If $\langle \xi, v \rangle < 0$ (v along Δ, like in pict), then $\{\varphi \geq 0\} \cap \Delta$ can be (for appropriate choice of φ) only the point $\{(0,0)\}$, so we have propagation.



If $\langle \xi, v \rangle > 0$, then we don't have propagation because
$R\Gamma_{\{\varphi \geq 0\}} k_{\mathbb{R}^2\Delta} \overset{\text{locally}}{=} k_{\mathbb{R}^2\Delta}$

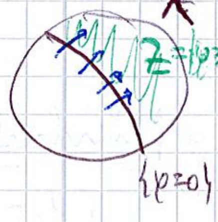


Conclusion: $SS(k_{\mathbb{R}^2\Delta}) \cap \pi^{-1}((0,0)) = \{\xi \in T^*_{(0,0)}\mathbb{R}^2 \mid \langle \xi, v \rangle \geq 0\}$ ($\pi : T^*\mathbb{R}^2 \to \mathbb{R}^2$)

(4) Half space

Take $\varphi : X \to \mathbb{R}$, with 0 regular value, then $Z := \{\varphi \geq 0\}$ and $F = k_{XZ}$

Then $SS(F) = \frac{1}{2}$ part of $T^*_{\{\varphi=0\}}X$ corresponding to "entering directions"



The proof is similar to before (ex 2-3)

- Properties

(a) $SS(F) \cap T^*_X X = \mathrm{Supp}(F)$ ($T^*_X X$ → zero set of T^*X)

(b) $SS(F)$ closed $\mathbb{R}^+$-conical subset of T^*X ($SS(F)$ is involutive)

(c) If $SS(F) = T^*_Z X$, $Z \subset X$ closed smooth, then F is loc isomorphic to k_{XZ} (in particular if $SS(F) = T^*_X X$ then F is locally constant)

10

Guiraux

(d) Triangular inequality:

If $F_1 \to F_2 \to F_3 \xrightarrow{+1}$ D.T. in $D^b(X)$,

Then $SS(F_i) \subseteq SS(F_j) \cup SS(F_k)$ for $\{i,j,k\} = \{1,2,3\}$

(e) If $F \in D^b(X)$, $f : X \to Y$ proper on $\mathrm{supp}(F)$,

Then $SS(Rf_*F) \subset f_\pi\left(f_t^{-1}(SS(F))\right)$

Def For $f : X \to Y$ smooth, we have a diagram

$$\begin{tikzcd} T^*X \arrow[dr, "\pi_X"'] & X \times_Y T^*Y \arrow[l, "f_t"' , "\text{given by } df"] \arrow[r, "f_\pi"] \arrow[d] & T^*Y \arrow[d, "\pi_Y"] \\ & X \arrow[r] & Y \end{tikzcd}$$

(f) Assume $SS(G)$ is non-characteristic for f

Then $SS(f^{-1}G) \subset f_t\left(f_\pi^{-1} SS(G)\right)$

Def $\Lambda \subset T^*Y$ is non characteristic for f

iff $f_\pi^{-1}(\Lambda) \cap T^*_X Y \subset X \times_Y T^*_Y Y$ ($\Leftrightarrow$ f_t is proper on $f_\pi^{-1}(\Lambda)$)

where $T^*_X Y = f_t^{-1}(0_X)$ — zero sect of T^*X

(if f is a closed embedding, we have transversality with X)

- Microlocal Morse Thm

Let X smooth mfld, $\varphi : X \to \mathbb{R}$, $F \in D^b(X)$

If φ is proper on $\mathrm{supp}(F)$ and $d\varphi_x \notin SS(F)$ if $\varphi(x) \in [a,b)$,

Then $R\Gamma(\varphi^{-1}((-\infty,b]);F) \xrightarrow{\sim} R\Gamma(\varphi^{-1}((-\infty,a]);F) \xrightarrow{\sim} R\Gamma(\varphi^{-1}((-\infty,a));F)$

~~Proof: Prove for $R\varphi_*F$~~

~~Since φ is proper on $\mathrm{supp}(F)$ & $SS(R\varphi_*F$~~

- <u>Microsupport of sheaves, 2nd part</u>

Guiraux, lect 3 (11)
30/06/16 Guiraux

Recall: X mfld, $F \in D(X)$, $h \in T^*X$

P_h = "propagation prop for F" holds iff $\forall \varphi$ smooth in a nbhd of $\pi(h)$

s.t. $\begin{cases} \varphi(\pi(h)) = 0 \\ d\varphi_{\pi(h)} = h \end{cases}$ Then $\left(R\Gamma_{\{\varphi \geq 0\}} F\right)_{\pi(h)} = 0$

$SS(F)^c$ = interior of $\{h \in T^*X \mid P_h \text{ holds}\}$

We stated some <u>basic</u> properties also. Let's now go back to functoriality.

If $f : X \to Y$, we can consider

$$\begin{tikzcd} T^*X \arrow[dr, "\pi_X"'] & X \times_Y T^*Y \arrow[l, "f_t"'] \arrow[r, "f_\pi"] \arrow[d] & T^*Y \arrow[d, "\pi_Y"] \\ & X \arrow[r, "f"'] & Y \end{tikzcd}$$

and denote $N_f^* = T_X^*Y := f_t^{-1}(0_X)$ zero sect. in T^*X

<u>Def</u> Λ is <u>non characteristic</u> for f if $f_\pi^{-1}(\Lambda) \cap T_X^*Y \subset X \times T_Y^*Y$

<u>Rmk</u> $T_Y^*Y = 0_Y$ = "zero set of T^*Y" by def

If f is a closed embedding, then
Λ is non charact. if $\Lambda|_X \cap T_X^*Y = \{0\}$

<u>Microlocal Morse lemma</u>: $\varphi : X \to \mathbb{R}$, $a < b$, $a, b \in \mathbb{R}$, $F \in D^b(X)$

If $\begin{cases} \varphi|_{\mathrm{supp}(X)} \text{ proper} \\ \forall x \text{ s.t. } \varphi(x) \in [a, b) \\ \quad \text{we have } d\varphi(x) \notin SS(F) \end{cases}$ Then $R\Gamma(\varphi^{-1}((-\infty, t)); F)$ is independent of t for $t \in [a, b)$

<u>Rmk</u> For $F = k_X$, we get classical Morse lemma

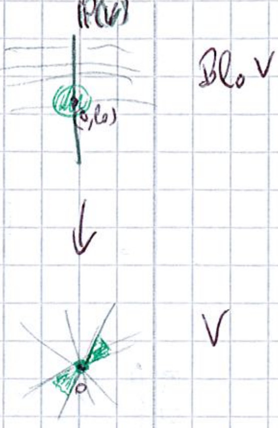
Example of application: $SS(F) = 0_X \subset T^*X \implies F$ is locally const.

- <u>Specialization of sheaves</u>

(a) <u>Real blowups</u>

Take V = vector space. Define $Bl_0 V := \{(v, \ell) \mid v \in V, \ell \in \mathbb{P}(V), v \in \ell\}$

We have map
$$\begin{array}{ccc} Bl_0 V & \ni & (v,l) \\ \downarrow & & \downarrow \\ V & \ni & v \end{array}$$
which is 1:1 outside fiber of 0, and preimage of 0 is $\mathbb{P}(V)$



Rmk If I take nbhd of $(0, l_0) \in Bl_0(V)$ and project it to V, I obtain a cone like in the right, which is not open in classic top. because of the origin

$\Rightarrow$ Using $Bl_0 V$ is just a "fancy" way to change the topology of V so that open sets are the cones! This is what microlocalization is about!

(b) General blowups

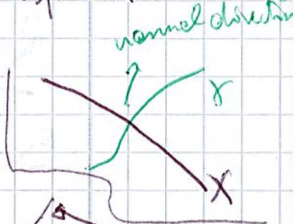
$W \subset V$, $V = W \oplus T$

$Bl_W V := \{(w, t, l) \in W \times T \times \mathbb{P}(T) \mid t \in l\}$

$$\begin{array}{ccc} Bl_X Y & \hookleftarrow & \mathbb{P}(T_X Y) \\ \downarrow & & \downarrow \\ Y & \longleftarrow & X \end{array}$$

- bijection above $Y \setminus X$
- above X, it is the total space of $\mathbb{P}(T_X Y)$

Def $Z \subseteq Y$ not included in X, the strict transform of Z is the closure of $Z \cap X^c$ in $Bl_X Y$

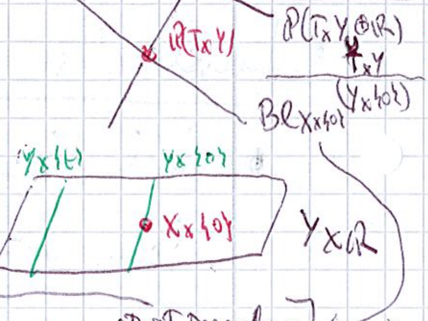


(c) Deformation to the normal cone

Take $X \subset Y$. What's $Bl_{X \times \{0\}}(Y \times \mathbb{R})$?

Normal bundle of $X \times \{0\}$ in $Y \times \mathbb{R}$ is $T_X Y \oplus \mathbb{R}$

Exceptional divisor is like in the picture on the right



(def. later!) [$Bl_{X \times \{0\}} Y \times \{0\}$ = strict transform of $Y \times \{0\}$]

Def deformation to the normal cone:

$$\tilde{Y}_X := Bl_{X\times\{0\}}(Y\times\mathbb{R}) \setminus Bl_{X\times\{0\}}(Y\times\{0\})$$

We have
$$\begin{array}{ccc} T_X Y \xrightarrow{i} \tilde{Y}_X & \hookleftarrow & \Omega \\ \downarrow & & \downarrow p \\ Y\times\mathbb{R} & \supset & Y\times\mathbb{R}^* \end{array}$$

. Above zero: $(\tilde{Y}_X)_0 = T_X Y \overset{i}{\hookrightarrow} \tilde{Y}_X$

Rmk in the picture, we saw $T_X Y \subset \mathbb{R}P(T_X Y \oplus \mathbb{R})$

Def $Z \subseteq Y$ not included in X, the <u>strict transform</u> of Z is the closure of $Z \cap X^c$ in $Bl_X Y$

(d) <u>Whitney cones</u>

$X \subset Y$, S subset of $Y \setminus X$

Def $C(S)$ cone of S along X is defined as $i^{-1}\left(\overline{p^{-1}(S\times\mathbb{R}^+)}\right)$

<u>Rmk</u> That's a closed cone, $\mathbb{R}^+$-invariant

In coordinates: (x', x'') coord on ~~Y~~ near $X = \{x''=0\}$ (so coord given by normal bundle)

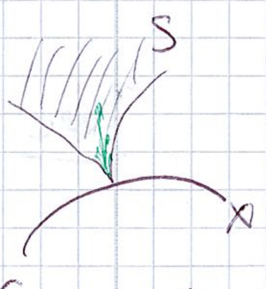
Then $v_x \in T_x Y$ belongs to $C_x S$

iff $\exists$ sequence (x_n', x_n'', t_n) s.t.

$t_n > 0$, $x_n'' \neq 0$, $(x_n', x_n'') \in S$

with $(x_n', x_n'') \xrightarrow{n\to\infty} x$

$t_n \cdot x_n'' \underset{n\to\infty}{\longrightarrow} v_x$



$C_{(0)}\gamma = \gamma$, γ cone

We have an analogue with pairs of sets $S_1, S_2 \subset X$ (X = given mfld):

Def $C(S_1, S_2) = C_{\Delta_X}(S_1, S_2)$ $\quad (T_{\Delta_X}(X\times X) \simeq TX)$

The description in coordinates is completely analogous ~~to~~ at the one of before.

• Involutivity Theorem

(14)

Guivaux

X, T^*X sympl. struct.

$S \subset T^*X$, $p \in S \Rightarrow$ two cones in $T_p(T^*X)$: (a) $C_p(S)$

(b) $C(S,S)_p \cap T_p(T^*X)$

Rf S is smooth $\overset{\text{closed}}{\vee}$, $C_p(S) = C(S,S)_p = T_pS$

Def $S \subseteq T^*X$ is <u>involutive</u> if $\forall p \in S$ $C(S,S)_p^{\perp} \subset C_p(S)$

Thm (K-S.) | $\forall F \in D^b(X)$, $SS(F)$ is a closed involutive subset of T^*X

$e^{k(y-x)}$ $e^{k(y-x)}$

$\triangle$ $C_p(S,S) \neq C_p(S)$ in general!

Example $X = \{0\} \subset Y = \mathbb{R}$, $S = [0,+\infty)$

$\Rightarrow$ $C_0(\Delta) = [0,+\infty) \subset T^*_X Y$

$C_0(\Delta,\Delta) = \mathbb{R} = T^*_X Y$

• Specialization of sheaves

F sheaf on Y.

$$\begin{tikzcd} T_X Y \arrow[r, hook, "i"] & \tilde{Y}_X \arrow[d] & \Omega \arrow[l, hook', "j"'] \arrow[d] \arrow[dr, "p"] & \\ & Y \times \mathbb{R} & Y \times \mathbb{R}^* \arrow[l, hook'] \arrow[r] & Y \end{tikzcd}$$

Def $\nu_X F := i^{-1} R j_* (p^{-1}(F))$ is

the <u>specialization</u> of F along X

Idea We want $\nu_X F$ to verify

$$(\nu_X F)_{v_x} = \varinjlim_{\substack{U \text{ "cone" in} \\ \text{the direct. } v}} F(U)$$