Sheaf quantization (Guillermou)

<u>aim</u> M manifold $\bar{\Lambda} \subset T^*M$ compact <u>exact</u> connected Lagr subm.

"exact": $\text{Liouville}|_{\bar{\Lambda}} = df \quad f : \bar{\Lambda} \longrightarrow \mathbb{R}$

$\Lambda \subset \dot{T}^*(M \times \mathbb{R}) = T^*(M \times \mathbb{R}) \setminus$ zero section

$\Lambda = \{ (x, t; \xi, \tau) \mid \tau > 0 \ , \ (x, \frac{\xi}{\tau}) \in \bar{\Lambda} \ , \ t = -f(x, \frac{\xi}{\tau}) \}$
$(x, \xi) \in T^*M \ , \ (t, \tau) \in T^*\mathbb{R}$

Λ <u>conic</u> Lagr subm

<u>Goal</u>: $\exists F \in D(k_{M \times \mathbb{R}})$, $k = \mathbb{Z}$, field

$\dot{SS}(F) = \Lambda \qquad (\dot{SS}(F) = SS(F) \setminus$ zero section$)$

We will deduce $\bar{\Lambda} \longrightarrow M$ homotopy equivalence

(Originally due to FSS 08, A, K, AK)

The idea to use $SS(F)$ in symp topology is due to Nadler, Zaslow and Tamarkin 08 (non displaceability using $SS(-)$).

Alternative construction of F by Viterbo, using Floer homology.

1) Examples of sheaves

(1) $B \subset \mathbb{R}^2$ open disk

$\Lambda_1 = T^*_{\partial B}\mathbb{R}^2$

The sheaves $F \in D(k_{\mathbb{R}^2})$, $\dot{SS}(F) = \Lambda_1$ with supp F compact

are $F = L_B$ for $L \in D(k)$.

idea: . $F_B \to F \to F_{\mathbb{R}^2 \setminus B} \xrightarrow{+1}$

$F_B = F \otimes k_B$

. $\dot{SS}(F_B) \subset \Lambda_1$

. triangle ineq $\rightsquigarrow$ $\dot{SS}(F_{\mathbb{R}^2\setminus B}) \subset \Lambda_1$

$\mathrm{supp}(F_{\mathbb{R}^2\setminus B}) = \partial B$

$i : \partial B \to \mathbb{R}^2$

$F_{\mathbb{R}^2\setminus B} = i_*(G)$, $G \in D(k_{\partial B})$

$SS(i_* G) = i_\pi\, i_t^{-1}(SS(G))$

with notation: if $f : M \to N$, $T^*M \xleftarrow{f_t} M \times_N T^*N \xrightarrow{f_\pi} T^*N$

$\Rightarrow G = 0$

$\Rightarrow F_B \simeq F$

$F_B \simeq j_! \underbrace{j^{-1} F}_{\text{constant}}$ where $j : B \hookrightarrow \mathbb{R}^2$

(2) $U = \mathbb{S}^1 \times]-\infty, 0[\ \subset \mathbb{S}^1 \times \mathbb{R}$

$\Lambda_2 = T^{*,\text{ext}}_{\partial U}(\mathbb{S}^1 \times \mathbb{R})$

$$\left\{ F,\ \dot{SS}(F) = \Lambda_2,\ \operatorname{Supp} F \subset \overline{U} \right\} \simeq \left\{ F \simeq p^{-1} L \otimes k,\ p : \mathbb{S}^1 \times \mathbb{R} \to \mathbb{S}^1,\ L \in D(k_{\mathbb{S}^1}),\ \text{locally cste} \right\}$$

Example : loc c^{st} sheaf.

- $k = \mathbb{C}$, $\alpha \in \mathbb{C}$, $L \in \mathcal{O}_{\mathbb{C}^*}$, $L(U) = \{ f \ ; (z\partial_z - \alpha)(f) = 0 \}$

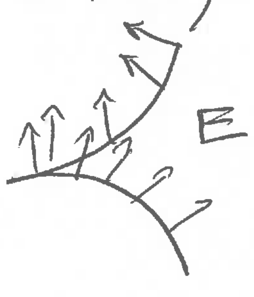
  $\rightsquigarrow$ monodromy $e^{2\pi i \alpha}$

- $k = \mathbb{R}$ loc syst on M = sheaf of horizontal sections of a vector bundle with a flat connection

If k ring $\{$ local syst on M with stalks V, V some k-mod$\}$

$\equiv \{$ representation of $\pi_1(M)$ in $V \}$

(3) $C \subset \mathbb{R}^2$, $C = \{ y^2 = x^3 \}$, Λ_3 = "positive half" of $\dot{T}^*_C \mathbb{R}^2$.

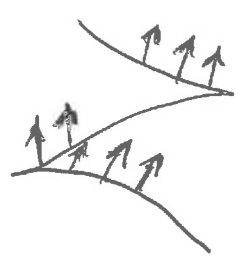


$\rightsquigarrow$ smooth Lag.

$E = \{ -x^{3/2} \leq y < x^{3/2} \}$

$$\{ F \in D(k_{\mathbb{R}^2}) \mid \dot{SS}(F) = \Lambda_3 \} = \{ L_{\mathbb{R}^2} \oplus L'_E \mid L, L' \in D(k) \}$$

④ $C' \subset \mathbb{R}^2$

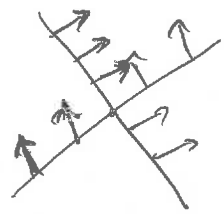


"Z-shape"

$\Lambda_4 =$ pos half of $\dot{T}^*_{C'}\mathbb{R}^2$

no $F \in D(k_{\mathbb{R}^2})$ with $\dot{SS}(F) = \Lambda_4$.

⑤ $D_\pm = \{x = \pm y\} \subset \mathbb{R}^2$ $\quad \Lambda_\pm =$ pos half of $\dot{T}^*_{D^\pm}\mathbb{R}^2$



$$\Lambda_5 = \Lambda_+ \cup \Lambda_-$$

Examples of sheaves with $SS \subset \Lambda_5$.

① $F_1 = k_{Z_+} \oplus k_{Z_-}$ $\quad Z_\pm = \{y \geq \pm x\}$

② $F_2 = 0 \to k_{\mathbb{R}^2} \longrightarrow k_{Z_+} \oplus k_{Z_-} \longrightarrow 0$

$$F_1 \to F_2 \longrightarrow k_{\mathbb{R}^2}[1] \xrightarrow{+1}$$

$\rightsquigarrow$ $\dot{SS}(F_2) = \Lambda_5$

$H^0(F_2) = k_{Z_+ \cap Z_-}$

$H^{-1}(F_2) = k_U$ $\quad U = \mathbb{R}^2 \setminus (Z_+ \cup Z_-)$

<u>Remarks</u> In examples ① & ② above $\Lambda_5 = \Lambda_5$ but in ② any F defined near $\pi(\Lambda_2)$ $\quad \pi : T^*M \to M$ with $\dot{SS}(F) = \Lambda_2$, can be extended (not in exp ①)

In ⑤, it is not easy to describe F near $\pi(\Lambda_5)$.

Kashiwara Schapira ⇝ makes sense to study F "near Λ"

2) The Kashiwara – Schapira stack

It is better to use "homotopy categories".

$Sh(k_M)$ dg – category of sheaves s.t. $[Sh(k_M)] = D(k_M)$

For example: Schnürer "6 Grothendieck operations"

($Sh(k_M)$ = dg- cat of "h-injective cplxs of injectives".

(We consider the bounded subcat)

$U \mapsto Sh(k_U)$ is a stack (= sheaf of categories)

$U \subset M$

Notation: • $F \in Sh(k_M)$ $\quad SS(F) = SS(\bar{F})$

$\bar{F} = F$ in $D(k_M)$

• For $S \subset T^*M$

$D_S(k_M) = \{F \in D(k_M) ; SS(F) \subset S\}$

$Sh_S(k_M)$

Definition: [KS] $D(k_M; S) = D(k_M) / D_{T^*M \setminus S}(k_M)$

and $Sh(k_M; S)$ = "Drinfeld quotient" (cf Toen "homotopy dg - category")

In $D(k_M; S)$ we invert the $u: F \to G$ s.t. $\mathrm{Cone}(u) \in D_{T^*M \setminus S}(k_M)$

In fact, all morphisms are of type $u = \underset{\text{any morph}}{v} \circ \underset{\text{cone}(s)}{s^{-1}}$

<u>Rk</u>. For $F \in sh(k_M; S)$, $SS(F) \cap S$ is well defined.

. For $S' \subset S$, $sh(k_M; S) \xrightarrow[\text{restriction}]{} sh(k_M; S')$

<u>Def</u>: . $\mu sh(k_{T^*M})$ = sheaf of dg-categories associated to $\left(\underset{\substack{\cap \text{ open} \\ T^*M}}{\Omega} \longmapsto sh(k_M; \Omega) \right)$

. For $\underset{\substack{\text{locally closed} \\ \text{subset}}}{\Lambda \subset T^*M}$ $\mu sh(k_\Lambda)$ subsheaf of $\mu sh(k_{T^*M})$

F s.t. $SS(F) \subset \Lambda$

<u>Quotient functor</u>

$$sh_{(\Lambda)}(k_M) = \{ F \in sh(k_M) \mid \exists \Omega \text{ nbhd of } \Lambda \text{ s.t. } SS(F) \cap \Omega \subset \Lambda \}$$

$$m_\Lambda : (sh_{(\Lambda)}(k_M))(U) \longrightarrow \mu sh(k_\Lambda)(T^*U \cap \Lambda)$$

<u>We will see</u>:

<u>Thm</u> $\bar{\Lambda} \subset T^*M$ compact exact connected Lagr,

then $\Lambda \subset \dot{T}^*(M \times \mathbb{R})$ conic Lagr.

For any $F \in \mu sh(k_\Lambda)$,

$\exists G \in sh_{\Lambda, 0_M}(k_{M \times \mathbb{R}})$, $G|_{M \times \{t\}} = 0$ $t \ll 0$

zero section.

and $m_\Lambda(G) = F$ in $[\mu sh(k_\Lambda)](\Lambda)$ ← homotopy cat

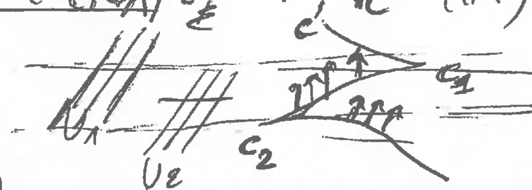
Moreover G is unique in $D(k_{M \times \mathbb{R}})$

Recall: For M mfd, $\mu Sh(k_{T^*M})$ sheaf of dg-cat on T^*M
associated to $\Omega \longmapsto Sh(k_M, \Omega)$
$$= Sh(k_M) / \{F \,/\, SS(F) \cap \Omega = \emptyset\}$$

$\Lambda \subset T^*M$, $\mu Sh(k_\Lambda)$ sh of cat on Λ

assoc with $\Omega \longmapsto Sh_\Lambda(k_M; \Omega)$

Rough idea of $\mu Sh(k_\Lambda)$: $\Lambda = T_c^*(\mathbb{R}^2)$, no object in $D_{\Lambda\cup 0}(k_{\mathbb{R}^2})$



but there are objects in $\mu Sh(k_\Lambda)$

E_1 = interior of cusp in U_1
$$\dot{SS}(k_{E_1}) = \Lambda \cap T^*(U_1)$$

E_2 = interior part of $c' \cap U_2$
$$\dot{SS}(k_{E_2}) = \Lambda \cap T^*U_2$$

$\Lambda_i := \Lambda \cap T^*U_i$

$\rightsquigarrow k_{E_i} \in \mu Sh(k_{\Lambda \cap T^*U_i})$

on $U_1 \cap U_2$,
$$k_{E_2 \cap U_1 \cap U_2} \longrightarrow k_{U_1 \cap U_2} \longrightarrow k_{E_1 \cap U_1 \cap U_2} \xrightarrow{\ s\ } k_{E_2 \cap U_1 \cap U_2}[1]$$

$\Rightarrow$ isom in $\mu Sh(k_{\Lambda_1 \cap \Lambda_2})$

$\rightsquigarrow$ glue k_{E_1} and $k_{E_2}[1]$ in $\mu Sh(k_\Lambda)$.

Hom spaces in μSh

Recall $\mu hom : D(k_M)^{op} \times D(k_M) \longrightarrow D(k_{T^*M})$

$$(F, G) \longmapsto \mu hom(F,G)$$

(i) $R\pi_* \, \mu hom(F,G) = R\mathcal{H}om(F,G)$ $\quad \pi : T^*M \to M$

(ii) $\mathrm{Supp}(\mu hom(F,G)) \subset SS(F) \cap SS(G)$.

in $D(k_M; S)$, $S \subset T^*M$ morph $F \xrightarrow{u} G' \xrightarrow{s} G$

(*) $SS(\mathrm{cone}(s)) \cap S = \emptyset$

$$\mathrm{Hom}_{D(k_M;S)}(F,G) = \varinjlim_{\substack{G \xrightarrow{s} G' \\ s.t.(*)}} \mathrm{Hom}_{D^b(k_M)}(F, G')$$

$$\xrightarrow{\sim} \varinjlim_{s_0} H^0(T^*M; \mu hom(F, G'))$$

$$\longrightarrow \varinjlim_{s_0} H^0(S; \mu hom(F, G')|_S)$$

$$\mu hom(F,G) \xrightarrow[\sim]{} \mu hom(F, G') \longrightarrow \underbrace{\mu hom(F, \mathrm{cone}(s_0))|_S}_{\substack{=0 \\ \text{by prop (ii)}}}$$

We get

$$\mathrm{Hom}_{D(k_M;S)}(F,G) \longrightarrow H^0(S; \mu hom(F,G)|_S)$$

Thm [KS] $S = \mathbb{R}_{>0} \cdot (x_0; \xi_0)$ $\quad (x_0, \xi_0) \in T^*M$

Then the above map is an isomorphism.

Corollary $F, G \in \mu Sh(T^*M)$, we have ~~$\mathcal{H}om(F,G)$~~

$\mathcal{H}om(F,G) \longrightarrow \mu hom(F,G)$ is an quasi-equiv ~~of categories~~.

$\quad \mathrm{Sh}(k_{T^*M}) \qquad \mathrm{Sh}(k_{T^*M})$

Recall: Thm: $SS(\mu hom(F,G)) \subset C_{\Delta_{T^*M}}(SS(F), SS(G))$

If Λ conic Lag, $SS(F), SS(G) \subset \Lambda$

then $\left.\begin{cases} SS(\mu hom(F,G)) \subset T^*_\Lambda(T^*M) \\ \mathrm{Supp}(\;''\;) \subset \Lambda \end{cases}\right\} \Rightarrow \mu hom(F,G)$ locally C^{st} on Λ

Def k field, $\dot{SS}(F) \subset \Lambda$ — smooth conic Lag

$\Big[$ F simple, $\mu hom(F,F) = k_\Lambda$

$id_F \longleftrightarrow 1$

Simple sheaves

$F \in D(k_M)$ $\quad SS(F) = $ closure of $\{(x;\xi) \in T^*M \mid m^\varphi_{(x;\xi)}(F) = (R\Gamma_{\{\varphi \geq 0\}}(F))_x \neq 0\}$

$\varphi : M \to \mathbb{R}$, $\varphi(x) = 0$, $d\varphi_x = \xi$.

Usually depends on φ.

Proposition (KS) $\Lambda \subset \dot{T}^*M$ smooth conic Lagr.

$p = (x_0; \xi_0) \in \Lambda$

φ_0, φ_1, $\varphi_0(x_0) = \varphi_1(x_0) = 0$, $d\varphi_{0,x_0} = \xi_0 = d\varphi_{1,x_0}$

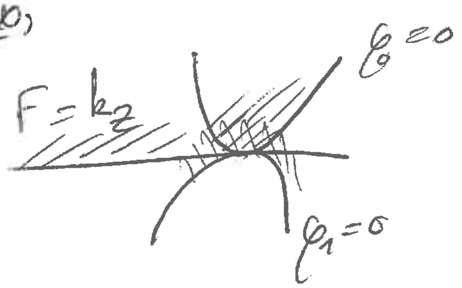
$\Lambda \pitchfork \Lambda_{\varphi_i}$ at p $\quad (\Lambda_{\varphi_i} = \text{graph } d\varphi_i)$, F s.t $SS(F) = \Lambda$

Then $m_p^{\varphi_1}(F) \simeq m_p^{\varphi_0}(F)\left[\dfrac{\tau_{\varphi_1} - \tau_{\varphi_0}}{2}\right]$

where $\tau_\varphi = \tau(T_p\pi^{-1}(x), T_p\Lambda, T_p\Lambda_\varphi)$

Maslov index

ex:



Define. • F is pure along Λ at p with shift $s \in \frac{1}{2}\mathbb{Z}$

if $m_\varphi(F)\left[s - \frac{1}{2}d_M - \frac{1}{2}\tau_\varphi\right]$ is in $d^\circ\, 0$
$\shortparallel L$

• F is simple if $L = k$

Ex:

shift changes by 1 at the cusps
"Maslov potential".

Proposition Λ as above; For any $p \in \Lambda$

(i) $\exists F \in D(k_M)$, $\Lambda_0 \subset \Lambda$ nbhd p
s.t. $SS(F) = \Lambda_0$ (in a nbhd of Λ_0)
F simple at all $q \in \Lambda_0$

(ii) For any $G \in D_{(\Lambda)}(k_M)$, for any $\Lambda_1 \subset \Lambda_0$ open contractible
we have $G \simeq F \otimes L_M$ in $\mu Sh(k_{\Lambda_1})$
where $L = \mu hom(F, G)_q \in D(k)$ $q \in \Lambda_1$

Cover $\Lambda = \bigcup \Lambda_i$, Λ_i open s.t. $\exists F_i \in D(k_M)$ as in prop (i)
then (ii) $\Rightarrow$ $F_i|_{\Lambda_{ij}} \simeq F_j|_{\Lambda_{ij}}[d_{ij}]$ in $\mu Sh(k_{\Lambda_{ij}})$ $\Lambda_{ij} = \Lambda_i \cap \Lambda_j$.

$\{d_{ij}\}_{ij}$ cocycle $d \in H^1(\Lambda, \mathbb{Z})$

Prop

d = Maslov class of Λ

Assume Maslov $= 0$, shifting F_i, we can assume

$F_i|_{\Lambda_{ij}} \simeq F_j|_{\Lambda_{ij}}$

$$s : \Lambda \longrightarrow \tfrac{1}{2}\mathbb{Z}$$
$$p \longmapsto \text{shift of } F_i \text{ at } p$$

$\mu Sh^s(k_\Lambda)$ = subcat of $\mu Sh(k_\Lambda)$ formed by the F — simple, shift $s(p)$ at p.

Prop $\mu Sh^s(k_\Lambda)$ is locally (in Λ) quasi-equiv to the category of local systems of rank 1 on Λ

Fact: Such categories are classified by $c_\Lambda \in H^2(\Lambda; k^*)$ (units in k.)

Thm $\forall F \in \mu Sh(k_\Lambda)$, $\overline{\Lambda} \subset T^*M$ cpct exact connected Lagr, $\Lambda \subset \dot{T}^*(M\times\mathbb{R})$ conic Lagr ($\subset \dot{T}^*(M\times\mathbb{R})$)

(i) $\exists G \in Sh(k_{M\times\mathbb{R}})$

$m_\Lambda(G) = F$, $G|_{M\times\{t\}} = 0$, $t \ll 0$

"quotient functor"

(ii) G is unique in $D(k_{M\times\mathbb{R}})$

idea (Tamarkin 08, KS's cutoff)

$D(k_{M\times\mathbb{R}}) \longrightarrow D(k_{M\times\mathbb{R}}, \{\tau > 0\})$ (t,τ) coord in $T^*\mathbb{R}$.

has an adjoint induced by

$$D(k_{M\times\mathbb{R}}) \longrightarrow D(k_{M\times\mathbb{R}})$$
$$F \longmapsto F * k_{[0,+\infty[}$$

We use a local version $F \longmapsto F * k_{[0,\varepsilon[}$

$\varepsilon \to 0$.

<u>Def</u> $U \subset M \times \mathbb{R}$

$$Sh^+(k_U) = \varinjlim_{V} Sh(k_V)$$

$V \subset M \times \mathbb{R} \times]0, \infty[$ $\quad$ $U \cup V$ nbhd of $U \times \{0\}$ in $M \times \mathbb{R} \times [0, +\infty[$

ε

$\gamma \subset \mathbb{R} \times]0, +\infty[$, $\gamma = \{(t,u),\ 0 \le t < u\}$

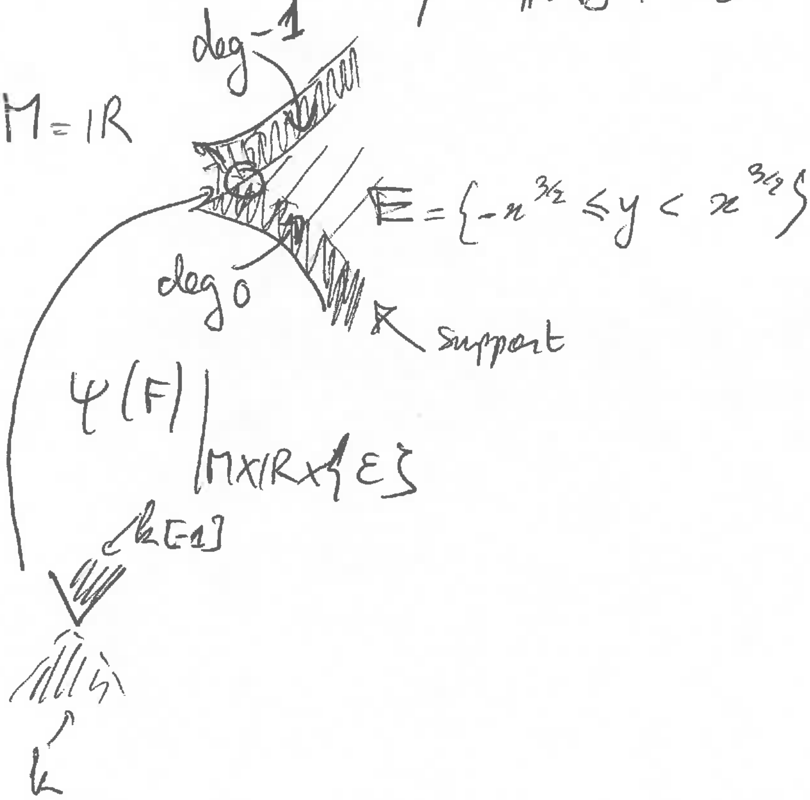
$$\Psi_U : Sh(k_U) \longrightarrow Sh^+(k_U)$$

$$F \longmapsto F * k_\gamma = s_!(q_1^{-1} F \otimes q_2^{-1} k_\gamma)$$

$$\begin{tikzcd} U \times \mathbb{R} \times]0,\infty[\arrow[r, "q_1"] \arrow[dr, "q_2"'] & U \\ & \mathbb{R} \times]0, +\infty[\end{tikzcd}$$

$s(x, t_1, t_2, u) = (x, t_1 + t_2, u)$

Ex $M = \mathbb{R}$



Recall: We've defined $\mu Sh(k_{T^*M})$, $\mu Sh(k_\Lambda)$, $\Lambda \subset \dot{T}^*M$ conic smooth lag

- Hom = μhom
- Notion of "simple sheaf" with shift $s \in \frac{1}{2}\mathbb{Z}$
- Maslov class $= 0 \iff \exists s : \Lambda \longrightarrow \frac{1}{2}\mathbb{Z}$ s.t. $\mu Sh^s(k_\Lambda) = F_{\text{simple}}$ shift $s(p)$ at p at $p \in \Lambda$

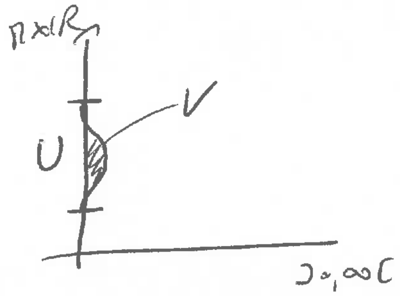
locally $\simeq Loc^1(\Lambda) =$ loc syst of rank 1.

There exists $c_\Lambda \in H^2(\Lambda; k^*)$ k^* invertibles

$c_\Lambda = 0 \iff \mu Sh^s(k_\Lambda)$ has global object.

Th $\bar{\Lambda} \subset T^*M$ compact exact connected Lagr. $\Lambda \subset \dot{T}^*(M \times \mathbb{R})$ conic smooth Lagr.
$\forall F$ global object of $\mu Sh(k_\Lambda)$
$\exists G \in Sh(k_{M\times\mathbb{R}})$, $m_\Lambda(G) \underset{q\text{-iso}}{\simeq} F$

- $U \subset M \times \mathbb{R}$, $Sh^+(k_U) = \varinjlim_V Sh(k_V)$.
  V runs over open sets in $U \times]0,+\infty[$
  $U \cup V$ nbhd of $U \times \{0\}$ in $U \times [0,+\infty[$



- $\Psi : Sh(k_U) \longrightarrow Sh^+(k_U)$
  $F \longmapsto F * k_\gamma$ $\quad \gamma = \{(t,u) \mid 0 \leq t < u\}$

Lemma If $F \in Sh(k_U)$ s.t. $\dot{SS}(F) = \emptyset$, then $\Psi(F) \underset{q\text{iso}}{\simeq} 0$.

Pf: base change $M = pt$
$$k_{\mathbb{R}} \times k_\gamma \big|_{u=\varepsilon} = Rs_!\left(k_{\mathbb{R}\times[q\varepsilon[}\right) = 0 \quad \square$$

Def $Sh^{pl}(k_{M\times\mathbb{R}})$ = sheaf of categories associated with
partially localized

$$U \longmapsto Sh(k_{M\times\mathbb{R}}; \dot{T}^*U)$$ on $M\times\mathbb{R}$

$\Rightarrow \Psi$ factorizes through $\Psi^{pl} : Sh^{pl}(k_U) \to Sh^+(k_U)$

"cut-off" ⇝ we can remove some parts of the microsupport

Given F, $p \in SS(F)$, $\exists F'$, $SS(F') \cap \mathrm{Cone} = SS(F) \cap \mathrm{Cone}$
$\mathrm{Cone} \ni p$ (pb near $\partial\mathrm{Cone}$)

Prop: $\Lambda \subset \dot{T}^*(M\times\mathbb{R})$ $\Lambda/\mathbb{R}_{>0} \to M\times\mathbb{R}$ finite

Then, $Sh^{pl}_\Lambda(k_{M\times\mathbb{R}}) \longrightarrow \pi_{\Lambda *}\,\mu Sh(k_\Lambda)$ $\pi_\Lambda : \Lambda \to M\times\mathbb{R}$

is a quasi-equivalence.

Pf: use cut-off functor to build a quasi-inverse □

Microsupport conditions

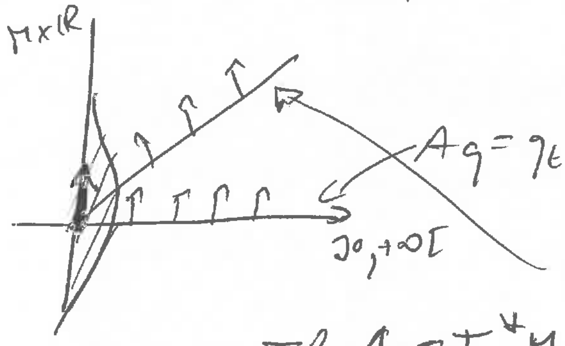
Prop: $q : M\times\mathbb{R}\times]0,\infty[\longrightarrow M\times\mathbb{R}$ proj

r: ——— $\longrightarrow$ ——— $r(x,t,u) = (x, t-u)$

$F \in Sh(k_U)$ distinguished triangle:
$U \subset M\times\mathbb{R}$

$$r^{-1}F[-1] \to \Psi(F) \to q^{-1}(F) \xrightarrow{+1}$$

$A \subset T^*(M\times\mathbb{R})$ $SS(F) \subset A \Rightarrow SS(\Psi(F)) \subset A_q \cup A_r$
(up to some limit)



$A_q = \pi_t\, q_\pi^{-1}(A) = A \times T^*_{\mathbb{R}}\mathbb{R}$

$A_r = \pi_t\, r_\pi^{-1}(A)$

If $A \subset T^*M \times \{\tau > 0\}$, $A_r \cap A_q = \emptyset$

Def: $Sh^+_A(k_U) = \varinjlim_V Sh_{A_q \cup A_r}(k_V)$
$A \subset T^*(M\times\mathbb{R})$
"$U \circ V$" neighb of U

$A \subset \{\tau > 0\}$, we have $Sh_{A_q \cup A_r}(k_V) \longrightarrow \mu Sh(k_{A_q}) \xleftarrow[\text{q. equiv.}]{\simeq} \mu Sh(k_A)$

$Sh_{(A_q)}(k_V)$ disjoint union

Deduce $m_A^+ : Sh_A^+(k_U) \longrightarrow \mu Sh(k_A)$

We have $m_A^+ \circ \Psi = m_A$

$\Psi : Sh_A(k_U) \to Sh_A^+(k_U)$

$m_A : Sh_A(k_U) \to \mu Sh(k_A)$

Recall: $\Psi = \Psi_U^{pl}$, $m_A^{pl} \circ \Psi^{pl} = m_A$

$\Psi^{pl} : Sh_A^{pl}(k_U) \longrightarrow Sh_A^+(k_U)$

Prop $\forall F \in \mu Sh(k_\Lambda)$, $\exists G \in Sh_\Lambda^+(k_{M\times\mathbb{R}})$ s.t. $m_\Lambda^+(G) = F$

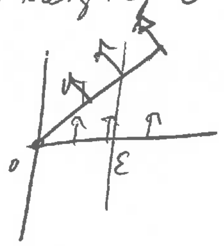
Proof: Choose $F_\Lambda \in Sh_\Lambda^{pl}(k_{M\times\mathbb{R}}) \simeq \pi_{\Lambda *}(\mu Sh(k_\Lambda))$

$$m_\Lambda(F_\Lambda) = F$$

$m_\Lambda^+ \circ \Psi^{pl}(F_\Lambda) = m_\Lambda(F_\Lambda) = F \qquad G := \Psi^{pl}(F_\Lambda)$

For ε small, $G_\varepsilon = G|_{M\times\mathbb{R}\times\{\varepsilon\}}$, G repr. by $G \in Sh(k_V)$

V neigh of U in $M\times\mathbb{R}\times\,]0,+\infty[$.



$\dot{SS}(G) \subset \Lambda_q \cup \Lambda_r$

$\dot{SS}(G_\varepsilon) = \Lambda \cup T_\varepsilon'(\Lambda)$

(We get twice ~~the~~ Λ as microsupport)

Pf of thm

Choose $\phi : \dot{T}^*(M\times\mathbb{R}) \times I \xrightarrow[\text{isot}]{} \dot{T}^*(M\times\mathbb{R})$

$\phi_s(\Lambda) = \Lambda$

$\phi_s(T_\varepsilon'(\Lambda)) = T_{\varepsilon+s}'(\Lambda) \qquad s \geq 0$.

Since Λ comes from $\overline{\Lambda} \subset T^*M$

$T_a'(\Lambda) \cap T_b'(\Lambda) = \emptyset$ if $a \neq b$

by [GKS] we make ϕ_s act on $D(k_{M\times\mathbb{R}})$, we obtain
$H_s \in D(k_{M\times\mathbb{R}})$, $SS(H_s) = \Lambda \cup T'_{\varepsilon+s}(\Lambda)$.

For $s \gg 0$, $\exists t_0 \in \mathbb{R}$

$\Lambda \subset T^*(M\times]-\infty, t_0[)$; $T_s'(\Lambda) \subset T^*(M\times]t_0+1, +\infty[)$
Then $G = f_*(H_s|_{M\times]-\infty, t_0+1[})$, $f:]-\infty, t_0+1[\xrightarrow{\sim} \mathbb{R}$
$f(t) = t$
$t < t_0$. $\square$

<u>Prop</u> $F \in D_\Lambda(k_{M\times\mathbb{R}})$, $F|_{M\times\{t\}} = 0 \quad t \ll 0$
$= F^+ \quad t \gg 0$
loc constant, indep of t.

If F' another such F
then $R\mathrm{Hom}(F, F') \simeq R\mathrm{Hom}(F_+, F'_+)$
In particular if $F_+ \simeq F'_+$ then $F \simeq F'$.

<u>Prop</u> Same hypothesis $R\mathrm{Hom}(F, F') = R\Gamma(T^*M, \mu hom(F, F'))$
$\xrightarrow[\text{restriction}]{} R\Gamma(\Lambda, \mu hom(F, F')|_\Lambda)$

To deal with Maslov class:

<u>Orbit category</u> (Keller)
$k' = k[\varepsilon]$, $\varepsilon^2 = 0$, $(k = \mathbb{Z}/2\mathbb{Z})$
$D(k_M) \to D_{[1]}(k_M) = D(k'_M)/P$ $\quad P =$ triangle subcat generated by $F \otimes_{k_M} k'_M$ for $F \in D(k_M)$

$k_M \to k'_M \to k_M \to k_M[1]$
$\overset{\shortparallel}{0}$ in $D_{[1]}(k_M)$ $\Rightarrow k_M \xrightarrow{\sim} k_M[1]$

For $F \in D(k_\Lambda)$, $i(F) \simeq i(F)[1]$

$\Rightarrow$ We can do the same thing in this category.

<u>Thm</u> $a : \Lambda \longrightarrow M$ projection

$\left[\pi_1(a) : \pi_1(\Lambda) \longrightarrow \pi_1(M)\right.$ is injective

<u>Proof:</u> • $k = \mathbb{Z}/2\mathbb{Z}$. Work in orbit category

$\rightsquigarrow$ we have $F_0 \in \mu sh(k_\Lambda)$ simple

$F \longrightarrow \mu hom(F_0, F)$ induces $\mu sh(k_\Lambda) \simeq Loc(k_\Lambda)$

$\mathcal{H}om_{\mu sh}(F_0, F)$ q.eq local systems

$G = \pi_1(\Lambda)$ $\rho : G \longrightarrow GL(k[G])$

regular representations (faithful)

$\rightsquigarrow L_\rho$ loc syst on Λ

$\rightsquigarrow F_\rho \in \mu sh(k_\Lambda)$ $\mu hom(F_0, F_\rho) = L_\rho$

<u>Thm</u>: $G_0, G_\rho \in Sh_\Lambda(k_{M\times\mathbb{R}})$ $m_\Lambda(G_0) = F_0$, $m_\Lambda(G_\rho) = F_\rho$

$L_0' = G_{0+}$, $L_\rho' = G_{\rho+}$ loc syst on M.

If we knew that $L_0' = G_{0+}$ is k_M, $p : M \times \mathbb{R} \longrightarrow M$

then $(G_0 \otimes p^{-1} L_\rho')_+ = L_0' \otimes L_\rho' = L_\rho'$

<u>Prop 1</u> $(G_\rho)_+ \simeq (G_0 \otimes p^{-1} L_\rho')_+ \Rightarrow G_\rho \simeq G_0 \otimes p^{-1} L_\rho'$

$m_\Lambda(*) \rightsquigarrow F_\rho = F_0 \otimes a^{-1} L_\rho'$

apply $\mu hom(F_0, -)$ $L_\rho \simeq a^{-1} L_\rho'$

Let $\rho_1 : \pi_1(M) \longrightarrow V$ repr assoc with L'_ρ. $\rho \cong a^* \rho_1$

$\rho|_{\ker \pi_1(a)} =$ trivial rep

ρ faithful $\Rightarrow$ $\ker \pi_1(a) = \{0\}$. $\cong$ $\square$

For general L'_ρ, do the same with $G_\rho \oplus \rho^{-1} L'_\rho$ and $G_\rho \oplus \rho^{-1} L_0'$

$\rightsquigarrow$ isom of representation.

Examples of quantization

(Vichery)

Plan

(I) what is a Quantization

(II) Examples for Lagrangians in T^*X (exact or not)

(III) Quantization of Ham isotopies (following Guillermou-Kashiwara-Schapira)

(I)

Question: $A \subset T^*X$ closed

Find $\mathcal{F} \in D^b(X)$ s.t. $SS(\mathcal{F}) = A$

$SS(\mathcal{F})$: conic, closed, coisotropic

Idea of Tamarkin, Pollesello-Schapira.

$$
\begin{tikzcd}
{}^*_{\tau>0}(X\times\mathbb{R}) := T^*(X\times\mathbb{R}) \cap \{\tau>0\} \arrow[rr, "\rho"] \arrow[dr, "\tilde{\rho}"'] & & T^*X \\
& J^1(X) \arrow[ur, "\pi"'] &
\end{tikzcd}
$$

$$\tilde{\rho}(x,p,t,\tau) = \left(x, \frac{p}{\tau}, t\right)$$

Def: • $\mathrm{Cone}(L) = \rho^{-1}(L)$. $L \subset T^*X$

• $\mathrm{Red}(A) = \rho(A \cap \tau>0)$ $\quad A \subset T^*(X\times\mathbb{R})$ conical.

(reduction $/\{\tau=1\}$).

Def A quantization of L is a sheaf $\mathcal{F} \in D^b(X\times\mathbb{R})$ s.t. $\mu S(\mathcal{F}) := \mathrm{Red}(SS(\mathcal{F})) = L$.

Tamarkin category

Tamarkin category

$D^b(X\times\mathbb{R}) / D^b_{\leq 0}(X\times\mathbb{R})$

{sheaves with SS in $\tau \leq 0$}.

$\mathcal{D}(X)$ full subcat such that $\mathcal{F} * k_{X\times[0,+\infty[} \simeq \mathcal{F}$.

$\mathcal{F} * \mathcal{G} = Rs_! \, \delta^{-1}(\mathcal{F} \boxtimes \mathcal{G})$

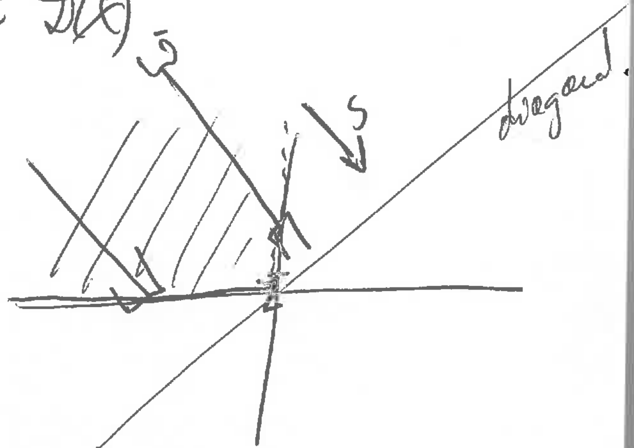
where $X\times\mathbb{R} \xleftarrow[\text{sum}]{s} X\times\mathbb{R}\times\mathbb{R} \xrightarrow{\delta} X\times\mathbb{R}\times X\times\mathbb{R}$

• $* k_{X\times[0,+\infty[}$ is a projector onto $\mathcal{D}(X)$.

If $X = \{pt\}$.

$k_{]-\infty,0[} * k_{[0,+\infty[}$

$= k_{[0,+\infty[}[-1]$



germ at x $= H_c(s^{-1}(x))$

II) Examples

• zero section ⇝ lift $\{(x,0,0)\}$ ⇝ wave front projection

$A = X\times\mathbb{R}_+$, $\mu S(k_A) =$ zero section

• Fiber

$\mu S(k_{pt\times[0,+\infty[}) = T^*_{pt}X$

• graph(df), $f \in C^2(X)$, front

$\mu S(k_{epi(f)}) =$ graph df

• $S : X \times \underbrace{\mathbb{R}^k}_{\substack{\text{quadratic} \\ \text{non-deg form at } \infty}} \longrightarrow \mathbb{R}$ generating function

$\rightsquigarrow L_S \subset T^*X$

Chaperon, Laudenbach – Sikorav: Ham-isotropic to 0-section

$\Rightarrow \exists S.$

$\pi : X \times \mathbb{R}^k \times \mathbb{R} \longrightarrow X \times \mathbb{R}$

$R\pi_! k_{epi(S)}$

<u>Proposition</u> If S is a g.f.q.i, $\mu S(R\pi_! k_{epi(S)}) = L_S$

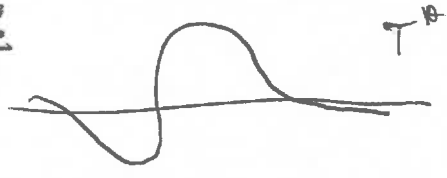
<u>Proof</u>: $SS(R\pi_! k_{epi(S)}) \subset \Lambda_\pi(SS(k_{epi(S)}))$

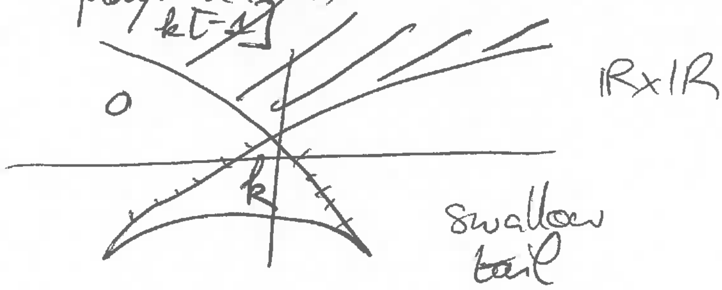
$$\Lambda_\pi = \{((x, \xi, t, x, t); (p, \underset{\frac{\partial S}{\partial \xi} = 0}{0}, \tau, -p, \tau))\}$$

$$\mu S(R\pi_! k_{epi(S)}) = L_S \qquad \square$$

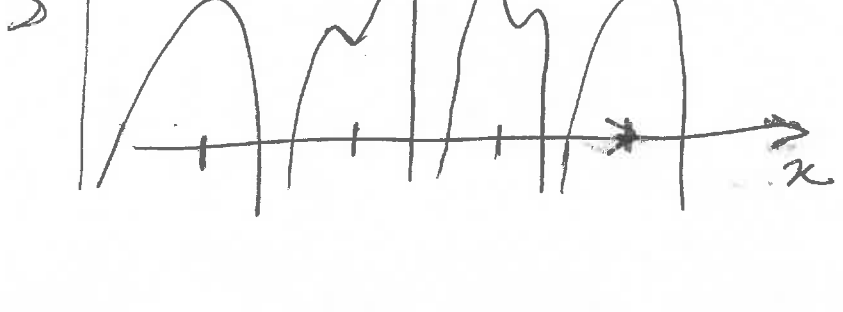
(The quadratic at ∞ condition plays a role)

<u>Ex:</u>





$S = -\xi^4 + \xi^2 - x\xi$



Thm (Guillermou)

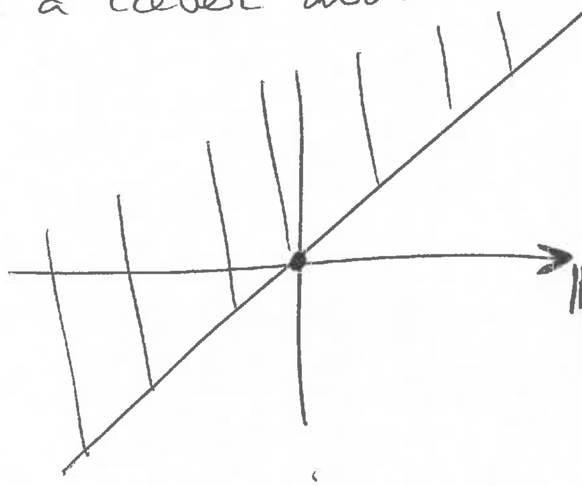
L exact compact in T^*X,
$\exists \mathcal{F} \in D^b(X \times \mathbb{R})$ such that $\mu S(\mathcal{F}) = L$
and for $t \gg 0$ this is a constant sheaf.
$t \ll 0$ this is 0.

what about L non-exact?

$L = \text{graph}(\eta)$, η is a closed one form

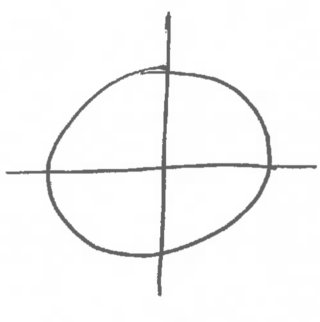
Ex: dq on S^1

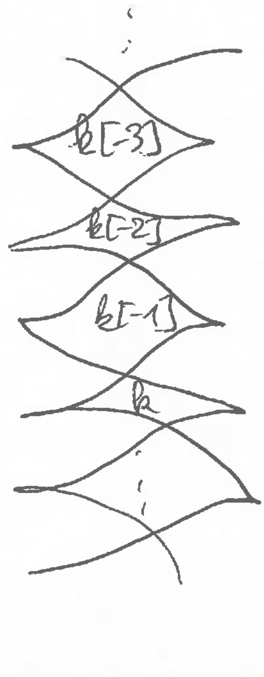
Lift to a cover and take epigraph



and reproject

Ex:





$\mu S(\mathcal{F}) =$ 

III Quantization of homogeneous Hamiltonian isotopies

[GKS]

$$\phi : \dot{T}^*X \times I \longrightarrow \dot{T}^*X$$

Homogeneous Hamiltonian

$$\Lambda_\phi = \{(\phi(t,\xi), \xi, t, -H_\phi(t, \phi(t,\xi)))\} \subset T^*(X \times X \times I)$$

Thm: ϕ is HHI, then there exists $K \in D^{b,loc}(X \times X \times I)$ such that

(i) $SS(K) \subset \bar{\Lambda} = \Lambda_\phi \cup T^*_{X\times X\times I}(X \times X \times I)$

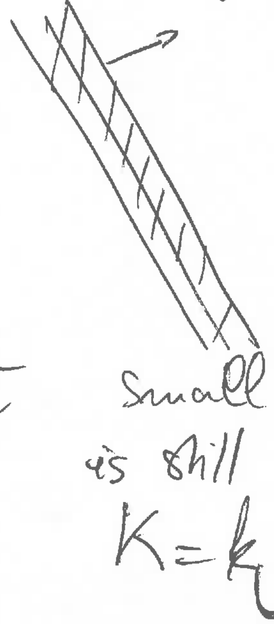
(ii) $K|_{t=0} = k_\Delta$

Step 1: Find quantization of a deformation of the diagonal

$$U = \{(x,y) \in X \times X \,,\, d(x,y) \leq \varepsilon\}$$

$\varepsilon \ll$ inj radius

$SS(k_U)$ is the graph of the geod flow at time ε



$(x,y,v,-v)$

s.t. $y = x + \frac{v}{|v|}$

$t \in]-\varepsilon, \varepsilon[$ Small perturbation $\phi_t(T^*_{\partial U}(M \times M \times I))$ is still the outward conormal to U_t

$$K = k_{\bigcup (U_t \times \{t\})}$$

$K = D^b(X \times X)$ Kernels

$$K^{-1} = v_*\left(R\mathcal{H}om(K, p_Y^! k_Y)\right) \qquad v(x,y) = (x,y)$$

$$k_U^{-1} = k_{\bar U}[-n] \qquad k_U^{-1} \circ k_U = k_\Delta$$

$$K = K' \circ k_{\bar U}[-n] \qquad \square\,?$$

Composition of kernels

$$\begin{tikzcd}
\overset{\mathcal{F}}{X \times Y} & & \overset{\mathcal{G}}{Y \times Z} \\
& X \times Y \times Z \arrow[ul, "p_{XY}"'] \arrow[ur, "p_{YZ}"] \arrow[d] & \\
& X \times Z &
\end{tikzcd}$$

$$\mathcal{F} \circ \mathcal{G} = Rp_{XZ!}\left(p_{XY}^{-1}\mathcal{F} \otimes p_{YZ}^{-1}\mathcal{G}\right)$$