Kontsevich I

Quantization and Fukaya category of complex symplectic manifolds.

Symplectic mfd : can be quantized in several different ways.

Symp mfd $\rightsquigarrow$ sheaf of abelian categories
/$\mathbb{R}$ (1) DG modules / $\mathbb{R}$ $[[\hbar]]$
/$\mathbb{C}$ $\mathbb{C}$

cf. deWilde – Lecompte
Fedosov
Maeda

Star product: $f * g = fg + \hbar\{f,g\} + \cdots$
associative. $+\hbar^n \cdot$ bidifferential operator in f, g, degree $\leq n$ in each f, g.

Thm $\exists$ canonical (locally) $*$ product, canonical up to inner automorphism, on $\mathcal{O}_{mfd}[[\hbar]]$.

canonical sheaf of categories of modules

$$\bigcup \otimes_{\mathbb{C}[[\hbar]]} \mathbb{C}((\hbar))$$

holomorphic modules

Serre. At $\hbar = 0$, dim support $= \frac{1}{2}$ dim M.

(2) $\rightsquigarrow$ Fukaya category.

Well defined if M is "convex at infinity".

(<u>Not</u> convex at ∞ : $\mathbb{R}^{2n} \setminus \{0\}$, $n \geq 2$).

Boundary conditions for objects at ∞.

Holonomic objects $/ \mathbb{C}[[e^{-1/\hbar}]]$ if $[\omega] \in H^2(X, \mathbb{Z})$

exp? $\mathbb{Z}\{e^{-1/\hbar}\}$, convergent if $|e^{-1/\hbar}| \ll 1$.

$(M, \omega = \omega^{2,0})$ algebraic symplectic mfd.

not formal in $\hbar$ quantization.

Suppose (M, ω) is convex at ∞ in algebraic sense:

$(M, \omega) \subset$ complex algebraic Poisson mfd $(\bar{M}, \alpha \in \Gamma(\Lambda^2 T\bar{M}))$

open dense sympl. leaf

$\alpha|_M = \omega^{-1}$.

Conditions: (? maybe too strong ?):

- $H^{2,0}(M, \mathbb{C}) = H^{2,0}(\bar{M}, \mathbb{C}) = 0$.
- $\bar{M} \setminus M$ is ample divisor, maybe singular.

$\rightsquigarrow$ Construct a canonical family of algebras

$\mathcal{O}_\hbar(M) / \mathbb{C}[[\hbar]]$, flat.

Filtrations $\mathcal{O}_\hbar^{\leq i}(M)$: deformation of $\mathcal{O}(M)^{\leq i}$, function w/ pole of order $\leq i$ at $\overline{M} \setminus M$.

Expected: / entire functions in $\hbar$.

algebraic / change of parametrization of $\hbar$.

$M = T^* X$ $\quad$ X affine variety $/ \mathbb{C}$.

$\mathcal{O}_\hbar(M)$ = differential operators in $K_X^{\otimes 1/2}$

$\hbar \neq 0$

[In local coords, roughly express in $\hbar$, $\hbar \frac{d}{dx}$, x (?)]

Assoc. Poisson mfd.

$X \subset \overline{X}$; $\overline{X} \setminus X$ division w/ normal crossings

$$T^*_{\log} \xrightarrow[\text{spaces}]{\text{affine}} \overline{X}$$

$\overline{M}$ = compactify as proj. space in fibres.

(dependence on $\hbar$ is algebraic)

rational case

$$M = (\mathbb{C}^*)^2 \quad , \quad \omega = \frac{dz_1}{z_1} \wedge \frac{dz_2}{z_2} .$$

Quantum generators: $\hat{z}_1^{\pm 1}$, $\hat{z}_2^{\pm 1}$

w/ relat$^\circ$: $\hat{z}_1 \hat{z}_2 = q \hat{z}_2 \hat{z}_1$

trigonometric case

where $q = \exp(\hbar) \in \mathbb{C}^*$.

Elliptic case

$M = \mathbb{C}P^2 \setminus$ cubic curve

$\omega = \dfrac{dz_1 \wedge dz_2}{\text{cubic poly}}$

$\sim$ "Skylenian type algebras"

quantum generators for algebra Skl $/\!/$ $\mathbb{C}\langle \hat{z}_i \rangle$ / relat below.

$\hat{z}_1, \hat{z}_2, \hat{z}_3$

s.t. • $[\hat{z}_1, \hat{z}_2] = \alpha \hat{z}_1 \hat{z}_2 + \beta \hat{z}_3^2$

• $\mathbb{Z}/3$ permutat of this.

$\mathbb{Z}_{\geq 0}$ graded.

Centre: $H = \gamma_1 \left(\hat{z}_1 \hat{z}_2 \hat{z}_3 + \hat{z}_2 \hat{z}_3 \hat{z}_1 + \hat{z}_3 \hat{z}_1 \hat{z}_2 \right)$

$+ \gamma_2 \left(\hat{z}_3 \hat{z}_2 \hat{z}_1 + \hat{z}_2 \hat{z}_1 \hat{z}_3 + \hat{z}_1 \hat{z}_3 \hat{z}_2 \right)$

$+ \gamma_3 \left(\hat{z}_1^3 + \hat{z}_2^3 + \hat{z}_3^3 \right)$

γ_i depend on α, β.

Consider the degree zero part of Skl $[H^{-1}]$ . .

Parametre space : $\mathcal{M}_{1,2}$.

$\left(\text{elliptic curve } E + \begin{matrix} \text{shift } E \to E \\ x \mapsto x + x_0 \end{matrix} \right)$

shift $= \exp\left(\hbar \, (\operatorname{Res} \omega)^{-1} \right)$

"in group law" : $T_E \to E$.

More on convexity at ∞:

- $\mathbb{C}^{2n} \setminus \{0\}$ $\quad (n \geq 0)$ is not algebraically convex.

- Also not convex: nilpotent coadjoint orbit in $(\mathfrak{sl}_n)^*$.

- Assume $(M, \omega) / \mathbb{C}$ algebraic symplectic, algebraically convex at ∞ (we can perhaps relax the previously given def, which he thinks is too restrictive).

$\rightsquigarrow$ Holomorphic family $/ \mathbb{C}_\hbar \setminus 0$ of categories

$\bigcup$

holonomic objects $C_\hbar^{hol}(M, 0)$.

A_∞ $\overset{\sim}{=}$?

Generalized Riemann-Hilbert correspondence!

- Consider M as real symp:

$$M, \underbrace{\mathrm{Re}\left(\frac{\omega}{\hbar}\right)}_{\text{real symp. form}} + \underbrace{i\, \mathrm{Im}\left(\frac{\omega}{\hbar}\right)}_{\text{treat as B-field, rep. class in } H^2(M, i\mathbb{R}/2\pi\mathbb{Z})}$$

Fukaya cat:

$$\underbrace{\mathrm{Hol}}_{\text{holonomic}}\ \mathcal{F}\left(M, \mathrm{Re}\left(\frac{\omega}{\hbar}\right) + i\,\mathrm{Im}\left(\frac{\omega}{\hbar}\right)\right)$$

(f. dim. morphism spaces?)

- Csq: would induce T structure on top side, ...

- This correspondence $\cong$ would be:
  - analytic in $\hbar$
  - $\mathbb{C}$-analytic in objects.

Let's see how this would work in some EXAMPLES.

- $M = T^* X$, w/ $\dim_{\mathbb{C}} X = 1$, X affine.

Twist both sides by $K_X^{\otimes 1/2}$.

holonomic $\mathcal{D}(X)$-modules ?

$\hbar \neq 0$

Classification :

$\mathcal{E} \in \mathrm{Hol}\ \mathcal{D}(X)$-mod.

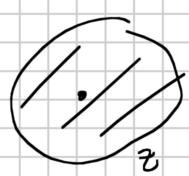
given by data of :

$\left\{ \begin{array}{l} \cdot \text{ finite set } \quad S \subset X \\ \cdot \text{ on } X \setminus S : \text{ algebraic vector bundle w/} \\ \qquad\qquad\qquad\qquad \text{connection } \nabla. \end{array} \right.$

data for ∇:

- monodromy $\pi_1(X \setminus S, x_0) \longrightarrow GL(n, \mathbb{C})$
- Stokes data at $S \cup S_\infty$, where $S_\infty := \overline{X} \setminus X$.

[Recall: <u>Stokes data</u>:



disc. coord. $\quad \mathcal{F} \big/ \mathbb{C}((z))$,

fin dim space

$$\nabla : \mathcal{F} \longrightarrow \mathcal{F} \otimes_{\mathbb{C}((z))} \mathbb{C}((z))\,dz$$

On components (?), $\mathcal{F}$ decomposes as

$$\mathcal{F} \underset{\text{canon.}}{\simeq} \bigoplus \mathcal{F}_{f_*},$$

where $f_\alpha \in \left(\bigcup_{k \geq 1} \mathbb{C}[z^{-1/k}] z^{-1/k} \right) \Big/ \hat{\mathbb{Z}}$

$$z^{-1/k} \longmapsto e^{-\frac{2\pi i}{k}} z^{-1/k}.$$

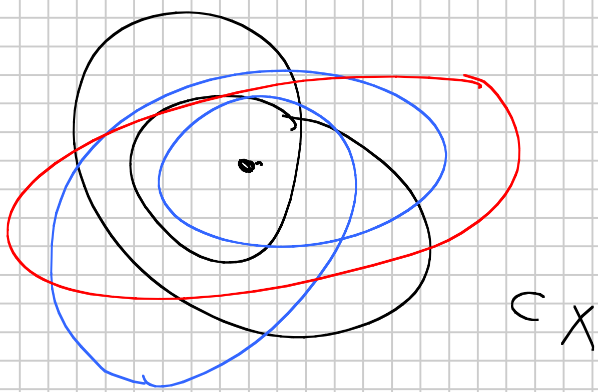
"Galois grp action"
(minimal possible k.)

$e^{f_\alpha(z)} \mathbb{C}((z^{1/k}))$ module $/ \mathbb{C}((z^{1/k}))$, ∇.

$$\mathcal{F}_{f_\alpha} = e^{f_\alpha(z)} \mathbb{C}((z^{1/k})) \underset{\mathbb{C}((z^{1/k}))}{\otimes} \text{module w/ regular singularities}$$

What is Stokes data?

At pt $x_0 \in S \cup S_\infty$, finite collection of f_α's.

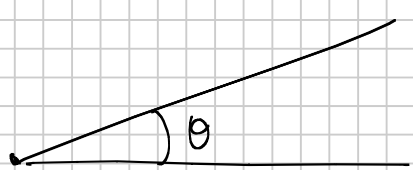


$f_\alpha \longleftrightarrow$ circle

winding number is the minimal k introduced above.

curves about one pt of S
(in local coords)

Any $z = \theta$ (ie a ray)



$e^{i\theta}$ $\exp\left(\mathrm{Re} \left(\text{branches } f_\alpha \right) \left(\varepsilon e^{i\theta} \right) \right)$

Choose $0 < \varepsilon << 1$.

$|e^{f_\alpha(z)}| \approx$ growth of solut° of diff. eq. (?).

Can think of the immersed circles as cooriented. ;

Consider constructible sheaves on X w/

$$SS \subset \left(\begin{array}{l} \bigcup \text{ positive conormal bundles to curves} \\ \bigcup \text{ zero\_section} \end{array} \right) \Big\backslash \begin{array}{l} \text{small} \\ \text{parts of} \\ \text{0-sect} \\ \text{near } S_\alpha \end{array}$$

ie the circles on X.

<u>Note</u>: You should add $f_\alpha = 0$, irregular, to the list of (Puiseaux) terms at every point.

Better way to speak about irregular terms (still for $\dim_{\mathbb{C}} M = 2$, eg $M = T^*X$, $\dim_{\mathbb{C}} X = 1$ – as above.)

$$\overline{M} \supset M$$

Poisson surface $\qquad$ open symplectic leaf.

$\exists$ blowup of $\overline{M}$ at $\overline{M} \setminus M$:

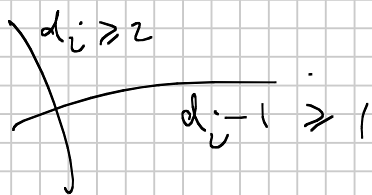
Recall $\overline{M} \setminus M$ = divisor w/ simple normal crossings

ω has pole of order $d_i \geq 1$ along C_i.

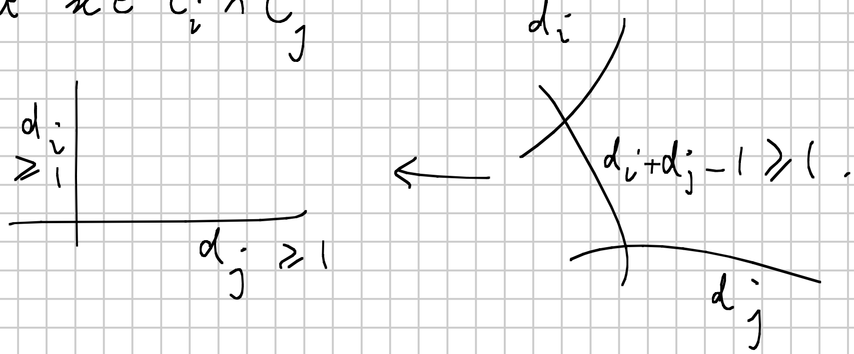
One can always blow up at

- pt $x_i \in C_i$, $x_i \notin C_j$ $(i \neq j)$

w/ $d_i \geq 2$, $\rightsquigarrow$



- pt $x \in C_i \cap C_j$

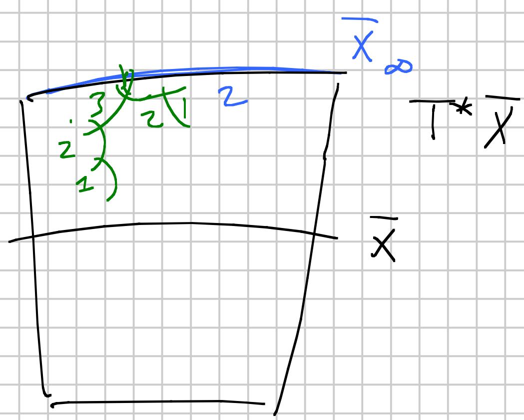


$$\bigsqcup_{x \in \overline{X}} \left\{ \begin{array}{c} \text{irregular terms} \\ \text{at } x \end{array} \right\}$$

$$= \varinjlim_{\substack{\text{Poisson compactifications} \\ \text{of } T^* \overline{X}}} \left(\text{log divisors } C_i : d_i = 1 \right)$$

(where $M = T^* \overline{X}$ now)

<u>Ex.</u> (for Vivek)



Repeatedly make blow-ups at ∞

((!) blow-up diagram could be wrong – I can't quite read it.)

$M \subset$ Poisson surface $\overline{M}$

$\dim M = 2$ $\qquad$ $\overline{M} \setminus M = \bigcup C_i$ normal crossings

$M = T^* \overline{X}$ $\qquad$ $\bigcup$

$\qquad \bigcup_{d_i = 1} C_i^\circ$ log part

$\rightsquigarrow$ Full sub-category of holonomic modules

In dim 2: automatically get symplectic convexity at ∞.

$\rightsquigarrow$ Version of the Fukaya category

$C_i^0 = \mathbb{C}$ $\qquad \partial(\text{tubular nhood}) = S^1 \times \underset{\mathbb{C}}{\overset{\cap}{\mathbb{R}^2}}$

Objects: Lag. branes s.t. $\partial = S^1 \times pt$

[something about seperating branes by using the flow on $\mathbb{R}^2$]

Ex. $X = \mathbb{C}^* \times \mathbb{C}^*$, w/ any toric compactification

Here all $C_i^0 = \mathbb{C}^*$.

$\{\text{irregular terms}\} = 2:1$ cover of $\mathbb{Q}P^1$

$= \{\text{primitive vectors in } \mathbb{Z}^2\}$

Riemann-Hilbert correspondance:

[Holonomic modules

$$\mathbb{C}\langle \hat{z}_1^{\pm 1}, \hat{z}_2^{\pm 1} \rangle \Big/ \hat{z}_1 \hat{z}_2 = q \hat{z}_2 \hat{z}_1 \qquad \text{if } 0 < |q| < 1$$
]

$$\wr\|$$

[Coherent sheaves / $\mathbb{C}^*/q^{\mathbb{Z}}$

$\mathcal{E}$ + two anti Harder-Narasimha filtrations.
]

Elliptic case ($X = \mathbb{CP}^2 \setminus$ cubic) ; can't do blowup — not sure what the story should be in this case.

<u>Rmk.</u> Log divisors work well for the $\dim_{\mathbb{C}} 2$ case.

Γ $z_1 = 0$:

$$\text{form } \frac{dz_1}{z_1} \wedge dz_2 + dz_3 \wedge dz_4 + \cdots$$

But what to use in higher dim?

Log divisors sufficient for T^*X, $(\mathbb{C}^*)^{2n}$, general toric varieties.

Questions:

- $\hbar \longrightarrow 0$ next time.
- He is using some sort of infinitesimal wrapping.