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Recall: $(M, \omega = \omega^{2,0})$ alg. sympl. mfld (eg. $M = T^*X$, $(\mathbb{C}^*)^{2n}, \ldots$)

convex at ∞ in $\mathbb{R}$-sympl. / alg. sense

$\rightsquigarrow$ 2 holom. families of A_∞-cat's over $\mathbb{C}_\hbar \setminus \{0\}$

1) $\mathrm{Fukaya}\left(\underset{\substack{\uparrow \\ \text{as } C^\infty\text{-mfld}}}{M}, \ \mathrm{Re}\left(\frac{\omega^{2,0}}{\hbar}\right) + i\,\mathrm{Im}\left(\frac{\omega^{2,0}}{\hbar}\right)\right)$ (? convergence for $0 < |\hbar| \ll 1$)

2) (semi) algebraic deformation quantization, considering only holonomic modules (dep^{ce} on $\hbar$ isn't algebraic)

eg. for T^*X, holonom. D-modules $\langle x_i, \hbar \frac{\partial}{\partial x_i} \rangle$

for $(\mathbb{C}^*)^{2n}$, modules $\langle \hat{x}_i^{\pm 1}, \hat{y}_i^{\pm 1} \rangle / \hat{y}_i \hat{x}_i = e^{\hbar} \hat{x}_i \hat{y}_i$

Conj (gen^d RH correspondence):

These categories are $\cong$ (and isom. is holomorphic wrt $\hbar$ in $|\hbar| \ll 1$).

- Natural limit as $\hbar \to 0$:

D-modules: $\left(\hbar \frac{d}{dx} + A_i(x)\right) \psi(x, \hbar) = 0$ $\quad A_i \in \mathrm{Mat}_{n \times n}(\mathbb{C}[x_1, \ldots x_n])$ (coords on X)

... as $\hbar \to 0$ get coherent sheaf on $M = T^*X$ whose support is algebraic Lagrangian (possibly singular) $L \subset M$ — the spectral curve.

Fix $L \subset M$, assume L smooth and assume in RHS, the support of the limiting module is $= L$ scheme-theoretically.

In alg. def quantization $\rightsquigarrow / \mathbb{C}[[\hbar]]$ get a holonomic sheaf of modules over a sheaf of algebras $/ \mathbb{C}[[\hbar]]$.

Thm: (Kashiwara-Schapira)

Modules in deform. quantiz$^{\underline{n}}$ which are flat $/ \mathbb{C}[[\hbar]]$, $\mathrm{supp}|_{\hbar=0}$ is L smooth without nilpotents

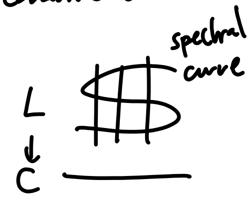
$\Longleftrightarrow$ Representations $\pi_1(L, x_0) \to GL(k, \mathbb{C}[[\hbar]])$

(up to twist by some class in $H^2(L, \mathbb{C}^*)$)

$\hookrightarrow$ "square root of canonical".

For $M = T^*C$, C curve, twist $=$ $\pi_1(L \setminus \text{ramif pts}) \to GL(k, \mathbb{C}[[\hbar]])$

st. $\circlearrowleft \mapsto -1$



Choose almost-HK structure $\omega^I, \omega^J, \omega^K$ real 2-forms ($d\omega^K \neq 0$) s.t. $\omega^{2,0} = \omega^I + i\omega^J$.

Lemma: if $\arg(\hbar) \neq \mathrm{Arg}\left(\int_{D \in \pi_2(M,L)} \omega^{2,0} \neq 0\right)$ then $\nexists$ pseudoholom. disc w/ boundary on L.
$H_2(M,L;\mathbb{Z})$

(indeed on such a disc we'd need to have pointwise $\frac{\omega^{2,0}}{\hbar}\Big|_{disc} \in \mathbb{R}_{>0}$)

$\rightsquigarrow$ Claim: for generic $\mathrm{Arg}(\hbar)$, $\exists$ canonical fully faithful embedding
$\mathrm{Reps}(\pi_1(L)) \hookrightarrow \mathcal{F}ukaya\left(M, \mathrm{Re}\left(\frac{\omega^{2,0}}{\hbar}\right) + i\,\mathrm{Im}\right)$

[consider L + loc. system as object of $\mathcal{F}(M)$]

[should choose a.c.s. compatible with $\mathrm{Re}(\omega^{2,0}/\hbar)$]

Rmk: as C^∞ manifold $\left(M, \mathrm{Re}\left(\frac{\omega^{2,0}}{\hbar}\right)\right)$ is CY: $c_1 = 0$

and L gives a real Lagrangian with Maslov class $= 0$.

Typically in such a situation, $\cup\{\partial$ of holom. discs w/ ∂ on $L\}$ has $\mathrm{codim}_{\mathbb{R}}\, 2$ in L. $\rightsquigarrow$ deformation of $\mathbb{Z}[\pi_1(L)]$

In our case $\mathrm{codim}_{\mathbb{R}}$ is 1! (because happens at discrete $\mathrm{Arg}(\hbar)$'s in 1-param. family).

- What happens if $\hbar \in$ Stokes ray?

As cross a Stokes ray, get a canonical automorphism of $\mathbb{Z}[\pi_1 L] \otimes \mathbb{Z}((e^{-1/\hbar}))^{\mathbb{R}}$

(wall-crossing transf. on moduli space of loc. systems) (= Novikov$_{\mathbb{Z}}$ terms)

of the form
$$\gamma_1 \mapsto \gamma_1$$
$$\gamma_2 \mapsto \gamma_2\left(1 + e^{-\frac{1}{\hbar}\omega(\circledcirc)}\gamma_1\right)$$

Conjecture: automorphisms $\in \mathbb{Z}[\pi_1(L)] \to \mathbb{Z}[\pi_1(L)] \otimes$ analytic functions

↳ this means bounds on growth of open GW invts.

Proposal: (RH correspondence at $0 < |\hbar| \ll 1$)

For each non-Stokes ray $e^{i\theta}\mathbb{R}_+$
holom. family of $\mathrm{Reps}(\pi_1(L))$ $\quad \xleftrightarrow{1:1} \quad$ holom. families$/_{0<|\hbar|\ll 1}$ of holonomic
$(\mathrm{Arg}(\hbar) - \theta) < \frac{\pi}{2}$ $\quad$ modules with lim suppt $\subset L$.

(asymptotic expansion / $\mathbb{C}[[\hbar]]$ as $\hbar \to 0$).

For some L, all Stokes automorphisms are trivial.

This happens eg when 1) $H_1(L,\mathbb{R})=0$

or 2) exact case $M=T^*X$, $\alpha = p\,dq$, $\alpha_{|L} = df$.
$d\alpha=\omega$

3) $M=T^*X$, $L=$ graph of closed 1-form. [is exact for modified Liouville form!]

given two such unobstructed objects (L_1,ρ_1), (L_2,ρ_2)

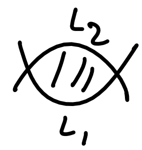
$\to$ canonical family of objects $\in \mathcal{F}(M, \omega^{2,0}/\hbar)$

Extend it to $\hbar=0$?

$\mathrm{Hom}(L_1,L_2) \simeq \mathbb{C}^{L_1\cap L_2}$

canonically for generic $\mathrm{Arg}\,\hbar$ there's no

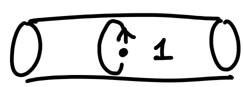
$\to$ Stokes rays for $L_1\cup L_2$?



- Ex: $X=\mathbb{C}^*$, $M=T^*\mathbb{C}^*$, $L_1 = \mathrm{graph}(0\,dz)$

$L_2 = \mathrm{graph}\left(\left(\frac{1}{z}-1\right)dz\right)$ ← non-exact

1 intersection at $z=1$, $Q=$ self-gradient flow lines from $z=1$ to itself?
of $\mathrm{Re}\left(\left(\frac{1}{z}-1\right)/\hbar\right)$.



2 Stokes rays for $L_1\cup L_2$ here.

F_- ($\mathrm{Re}\,\hbar<0$) | F_+ ($\mathrm{Re}\,\hbar>0$) in $t=\frac{1}{\hbar}$,

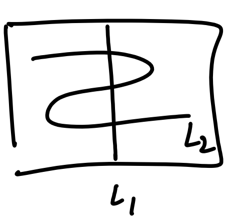
$F_-(\hbar)=F_+(-\hbar)$ | $F_+(\hbar=t^{-1}) = \dfrac{\Gamma(t)}{t^t e^{-t}}$

For $\hbar\in i\mathbb{R}_+$, $F_-(\hbar) = F_+(\hbar)\left(1+e^{-2\pi i/\hbar}\right)$
$\hbar\in i\mathbb{R}_-$ $F_-(\hbar) = F_+(\hbar)\left(1+e^{+2\pi i/\hbar}\right)$

from Γ function identities.

(NB: $\beta = d(\log z - z)$ so integrals in Floer theory are $\int e^{(\log z - z)/\hbar}\, dz/z$)
... hence Γ-function after chg. of variables...

- Ex:

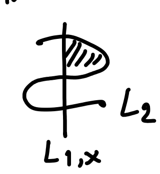


L_2 exact $\subset T^*X$

$L_1 = L_{1,x} = T^*_x X$ cotangent fiber

get walls in $\mathbb{C}^*_\hbar \times X_x$ where $\exists$ holom. disks

In particular at $\hbar=1$, see walls in X.



Outside of walls, $\mathrm{Hom}(L_{1x}, L_2) \simeq L_{1x} \cap L_2$

Say $\quad X = \mathbb{C}, \quad L_2 = d(\text{ alg. function } \varphi) \quad$ (multivalued)

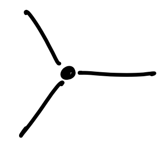
At branch pts $\quad L_2 \sim \mathrm{graph}(\pm z^{1/2} dz) \rightsquigarrow \varphi = \pm z^{3/2}$.

walls $=$ where two branches satisfy $\mathrm{Im}(\varphi_i) = \mathrm{Im}(\varphi_j)$

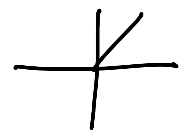
(then $\exists$ holom. strip between branches i & j following $\nabla \mathrm{Re}(\varphi_i - \varphi_j)$ closing up at branch pts)

loc. near simple branch pt, $\varphi = z^{3/2}$ real along 3 branches

$\Rightarrow$ graph of walls trivalent @ branch pts.



then propagate and scatter



...

- What happens if L is non-smooth or with nilpotents?

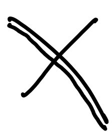
$\rightarrow$ more complicated asymptotics

for smooth L, $\quad \left(\hbar \frac{d}{dx} - A(x)\right)\psi(x,\hbar)$ sol^{n}s have $\psi(x,\hbar) = e^{S(x)/\hbar + \dots}$

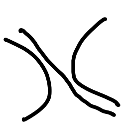
$L = \mathrm{graph}\ dS$

but here can also have eg. $e^{1/\sqrt{\hbar}}$ asymptotics.

<u>Hope</u>: sing. Lags's with infinitesimal movement



$L = L_0 \rightsquigarrow L_\hbar =$



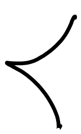
w/ $\displaystyle \frac{1}{\hbar}\int_{D_\hbar} \omega^{2,0} = \sum c_{a/b} \hbar^{-a/b}$

vanishing disc

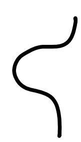
finite sum for $0 < \frac{a}{b} < 1$

$+ O(\log \hbar)$

Ex.



$L_0 : z_1^2 = z_2^3 \rightsquigarrow L_\hbar =$



$z_1^2 = z_2^3 + \hbar$

$z_2 \sim \hbar^{1/3}, \ z_1 \sim \hbar^{1/2}$

$\rightsquigarrow \displaystyle \frac{1}{\hbar}\int dz_1 dz_2 \sim \hbar^{-1/6}$.

Thm: $L_\hbar$ holom. family of sing. Lags. in $0 \le |\hbar| \ll 1$

$+$ local system of $S^1_{\mathrm{Arg}\,\hbar}$.

$\hookrightarrow$ global Fukaya.

Local Fukaya of $L_{\hbar=0}$

Stokes autom's now have fractional powers of $\hbar$.