Maksim Maydanskiy complementing to Sheridan's

Topics: 1. Floer cohomology as Morse theory on path space

2. Example of Gromov bubbling of closed hol. spheres

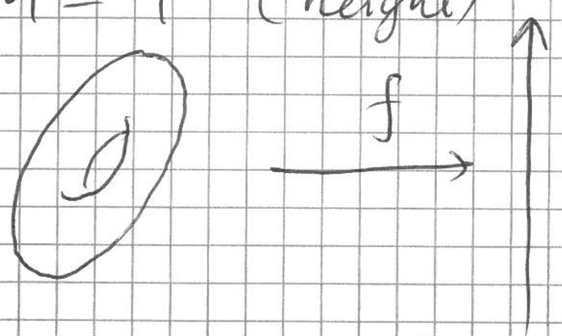
3. Example: computation of HF in $M = \mathbb{R}^2$ (exact Lag.s)

4. Floer cohomology of 0-section in T^*N vs Morse cohomology on N

§ 0 Morse theory

M smooth mfd. $f: M \to \mathbb{R}$ is Morse if for $p \in \mathrm{crit}(f)$, $\mathrm{Hess}_p f$ is non-deg.

Ex: $M = T^2$ (height)



Pick a metric g on M s.t. (g, f) is Morse-Smale (generic)

To this, associate Morse-Witten complex ($K = \mathbb{Z}_2$)

$$CM_*(M, f, g) = \bigoplus_{p \in \mathrm{Crit}(f)} \mathbb{Z}_2 \langle p \rangle$$

Differential: coefficient of q in ∂p is # solutions

$\gamma: \mathbb{R} \to M$, $\dot{\gamma} = \nabla_g f$, $\lim_{s \to +\infty} \gamma(s) = q$, $\lim_{s \to -\infty} \gamma(s) = p$.

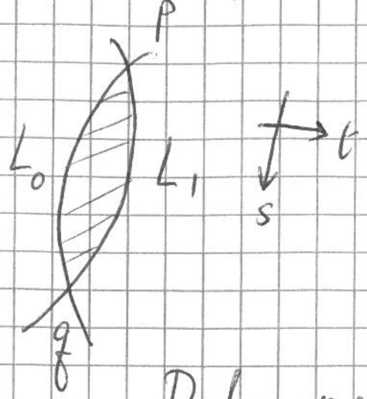
$\deg p$ = # negative eigenvalue of $\mathrm{Hess}_p f$.

Prop: $M(p, q)$ = mfd of dim($\deg p - \deg q$)

$M(p,q)/\mathbb{R}$ mfd of dim $\deg p - \deg q - 1$.

Thm: $HM(M) = H(M; \mathbb{Z}_2)$

§1. Floer cohomology as Morse theory



Idea: a strip $u(s,t)$ $(s\in\mathbb{R},\ t\in[0,1])$ is a trajectory of paths $u(s,\cdot):[0,1]\to M$ with end pt condition.

Def: path space $\mathcal{P}=\left\{\gamma:[0,1]\xrightarrow{C^\infty} M \text{ with } \begin{matrix}\gamma(0)\in L_0\\ \gamma(1)\in L_1\end{matrix}\right\}$ (∞-dim)

Then $T_\gamma\mathcal{P}=\left\{\eta\in\Gamma(\gamma^*TM) \quad \begin{matrix}\eta(0)\in T_{\gamma(0)}L_0\\ \eta(1)\in T_{\gamma(1)}L_1\end{matrix}\right\}$

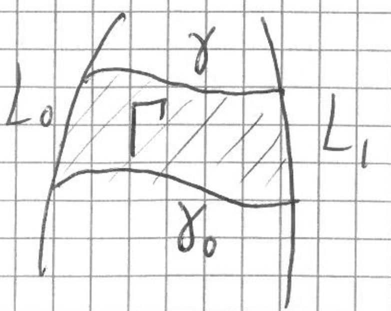
vector fields along γ

"Morse function"

$\mathcal{A}$ – action, not on $\mathcal{P}$ but on $\tilde{\mathcal{P}}$.

$\mathcal{A}(\gamma,\Gamma)=-\int_\Gamma\omega$

$\tilde{\mathcal{P}}=\{(\gamma,\Gamma)\mid\gamma\in\mathcal{P},\ \Gamma$ isotopy of γ to a fixed path γ_0 in the component of $\mathcal{P}$ specified by $\gamma\}$



"g" on path space

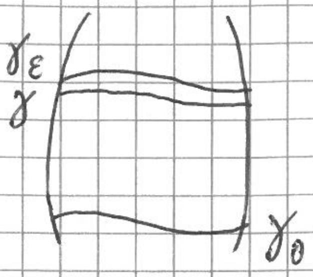
$\eta,\xi\in T_\gamma\mathcal{P}$.

$\underline{g}(\eta,\xi)=\int_0^1 g(\eta(t),\xi(t))\,dt$

compatible (with J_t) metric on X

Critical pts of $\mathcal{A}$:

consider $d\mathcal{A}_\gamma(\eta) = \int_0^1 \omega(\dot{\gamma}(t), \eta(t))$



$$\frac{\mathcal{A}(\gamma_\varepsilon) - \mathcal{A}(\gamma)}{\varepsilon} = \frac{\int \omega(\text{strip})}{\varepsilon}$$

Crit pts: pts where $d\mathcal{A}_\gamma(\eta) = 0$ for all η.

$\iff \dot{\gamma} \equiv 0$.

γ const $\quad \gamma(0) = L_0,\ \gamma(1) = L_1 \Rightarrow \gamma \in L_0 \cap L_1$

Gradient flow lines?

$$d\mathcal{A}_\gamma(\eta) = \int_0^1 \omega(\dot{\gamma}(t), \eta(t))\,dt \qquad \omega(\cdot,\cdot) = g(J\cdot,\cdot)$$

$$= \int_0^1 g(J\dot{\gamma}, \eta)\,dt = \underline{\underline{g}}(J\dot{\gamma}, \eta).$$

$u: \mathbb{R} \to \tilde{\mathcal{P}}$ a path in $\tilde{\mathcal{P}}$ $\quad u(s,t) = u(s)(t)$

the above is $\nabla\mathcal{A}_\gamma = J\frac{\partial u}{\partial t}$.

The flow line equation is

$$J\frac{\partial u}{\partial t} = \nabla\mathcal{A}_\gamma = u'(s) = \frac{\partial u}{\partial s} \quad \text{(hol. fun)}$$

§ 2 Floer cohomology of 0-section vs Morse cohomology

Floer	Morse
T^*N, 0-section $= L_0$	N
$L_1 = \text{graph}(df) \subset T^*N$	$f : N \to \mathbb{R}$
$df = 0 \Leftrightarrow$ pts in $L_0 \cap L_1$	crit pts of f
$L_0 \pitchfork L_1$	f Morse $\text{Hess}_p f$ non-deg

Hol strips:

$u : \mathbb{R} \times [0,1] \longrightarrow T^*N$

$u(s,0) \in L_0$

$u(s,1) \in L_1$

$u : \frac{\partial u}{\partial s} = J \frac{\partial u}{\partial t}$

<u>Claim</u>: $H = f \circ \pi$, for $\pi : T^*N \to N$ is s.t.

$\varphi^1(L_0) = L_1$.

Let $\tilde{u}(s,t) = \varphi_t^{-1}(u(s,t))$

If u satisfies $\frac{\partial u}{\partial s} = J \frac{\partial u}{\partial t}$,

then $\tilde{u}$ satisfies
$$\begin{cases} \frac{\partial \tilde{u}}{\partial s} = J\left(\frac{\partial \tilde{u}}{\partial t} - X_{H_t}\right) & (*) \\ \tilde{u}(s,0) \in L_0 \\ \tilde{u}(s,1) \in \varphi_1^{-1}(L_1) = L_0. \end{cases}$$

<u>Claim 1</u>: for $\gamma(s)$ Morse traj. of f

$\tilde{u}(s,t) = \gamma(s)$ satisfies $(*)$.

<u>Claim 2</u>: conversely, given $u \longrightarrow \tilde{u}$

$\longrightarrow$ obtain traj. γ which is a grad. traj. for f.