9:20 am

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# 1. Lagrangian Floer cohomology

## 1.1 Arnol'd conjecture

$(M, \omega)$ compact symp mfld

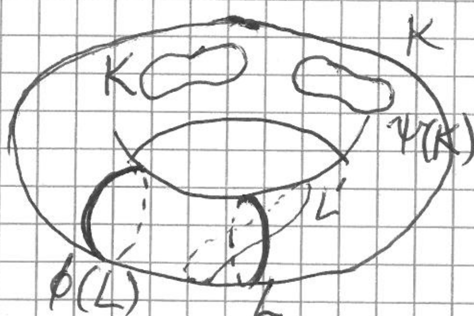
$L \subset M$ Lagrangian submfld

$\psi \in Ham(M,\omega)$ (i.e. $\psi = \psi_1$, $\psi_t$ = flow of $X_t$, $\omega(\cdot, X_t) = dH_t$)

Say $L'$ is Ham isotopic to $L$ if $\exists\, \psi \in Ham(M,\omega)$ s.t. $L' = \psi(L)$

Thm (Floer) If $L' \pitchfork L$, $L'$ Ham isot. to $L$, and $\omega|_{\pi_2(M,L)} = 0$, then $|L \cap L'| \geq \sum_i \dim H^i(L; \mathbb{Z}_2)$.

E.g.




K fails the hypothesis of the theorem.

$\phi \in Symp(M)$
$\phi \notin Ham(M)$

Floer associated to $L_0, L_1$ (transverse Lag. submflds with $\omega|_{\pi_2(M,L_i)} = 0$) a cochain complex $(CF(L_0, L_1), \partial)$ with cohomology $HF(L_0, L_1)$, such that:

1) $L_i'$ Ham isot to $L_i$ $(i=0,1)$ $\Rightarrow HF(L_0, L_1) \cong HF(L_0', L_1')$ (Ham isot invariance)
2) $CF(L_0, L_1)$ is a $\Lambda$-vector space with basis in bijection with $L_0 \cap L_1$
   ($\Lambda$: field extension of $\mathbb{Z}_2$)
3) $HF(L, L) \cong H^\cdot(L; \Lambda)$.

1) – 3) suffice to prove Floer's theorem.

## 1.2 The definition

$\mathbb{K}$ – field

$$\Lambda_{\mathbb{K}} = \left\{ \sum_{i=0}^{\infty} a_i T^{\lambda_i} : a_i \in \mathbb{K},\ \lambda_i \in \mathbb{R},\ \lim_{i\to\infty} \lambda_i = +\infty \right\}$$

Novikov field $/\mathbb{K}$

$\Lambda := \Lambda_{\mathbb{Z}_2}$

Define $CF(L_0, L_1) := \Lambda \langle L_0 \cap L_1 \rangle$

$$\partial p = \sum_{\substack{q \in L_0 \cap L_1 \\ \beta}} \# \mathcal{M}(p, q, \beta, J) \underbrace{T^{\omega(\beta)}}_{\Lambda} q$$

$\mathcal{M}(\dots)$ depends on $J$ = an $\omega$-compatible almost-cpx structure $J \in \mathrm{End}(TM)$, $J^2 = -Id$, $\omega(\cdot, J\cdot)$ = Riem. metric.

$\widehat{\mathcal{M}}(p, q, \beta, J) :=$ the set of <u>$J$-holomorphic strips</u>:

there are smooth maps $u : \underset{s}{\mathbb{R}} \times \underset{t}{[0,1]} \to M$

such that $\frac{\partial u}{\partial s} + J(u) \frac{\partial u}{\partial t} = 0$ ($J$-hol. curve eqn)

$u(s, 0) \in L_0$, $u(s, 1) \in L_1$

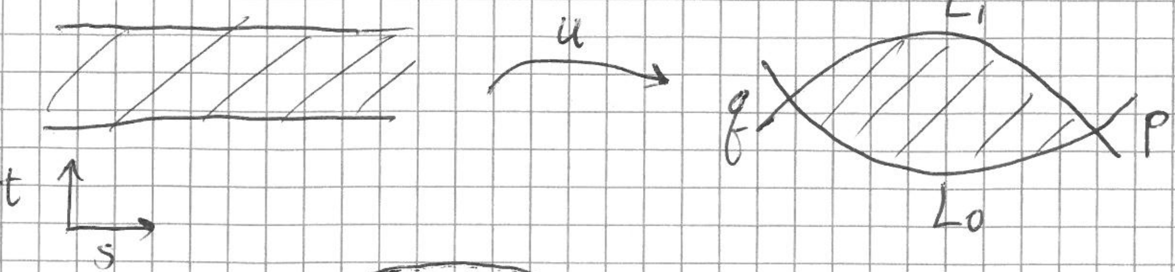
$\lim_{s \to +\infty} u(s, t) = p$, $\lim_{s \to -\infty} u(s,t) = q$, $[u] = \beta$.

$$\omega(\beta) = \int_{\mathbb{R} \times [0,1]} u^* \omega \underset{ex.}{=} \iint \underbrace{\left| \frac{\partial u}{\partial s} \right|^2}_{\omega\left(\frac{\partial u}{\partial s}, J \frac{\partial u}{\partial s}\right)} ds\, dt \geqslant 0 \qquad \text{energy}$$

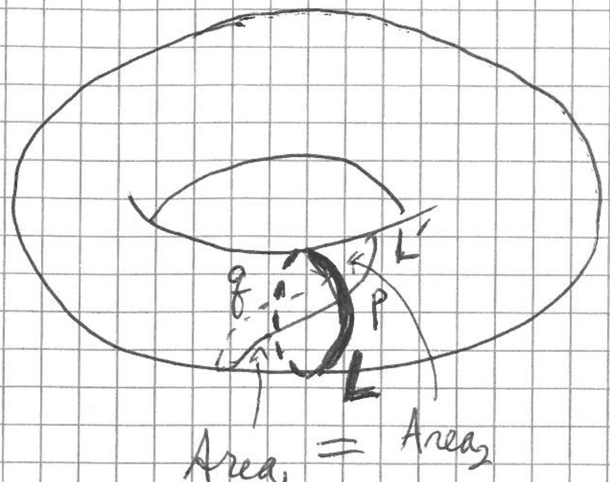
with equality iff $u$ is <u>constant</u>.

$\mathbb{R}$ acts on $\widehat{\mathcal{M}}(\dots)$ by <u>translation</u> $a \cdot u(s,t) := u(s+a, t)$

$\mathcal{M}(p, q, \beta, J) := \widehat{\mathcal{M}}(p, q, \beta, J) / \mathbb{R}$




E.g.




$CF(L, L') := \Lambda\langle p, q\rangle$

$\partial p = (T^{Area_1} + T^{Area_2})\, q$

If $A_1 = A_2$, $T^{A_1} + T^{A_2} = 0$

$\Rightarrow \partial = 0$

$\Rightarrow HF(L, L') \cong \Lambda\langle p, q\rangle$

If $A_1 \neq A_2$, WLOG $A_1 < A_2$,

then $(T^{A_1} + T^{A_2})^{-1} = T^{-A_1}(1 + T^{A_2 - A_1} + T^{2(A_2 - A_1)} + \dots)$

$\Rightarrow HF(L, L') \cong 0.$

## 1.3 Transversality

The defining PDE is a Fredholm problem.

When $J$ is regular, $\hat{M}(p, q, \beta; J)$ has structure of an $i(\beta)$-dimensional manifold.

Would like $J$ is 'generically' regular.

Need to modify to make $J \rightsquigarrow J_t$, $t \in [0, 1]$

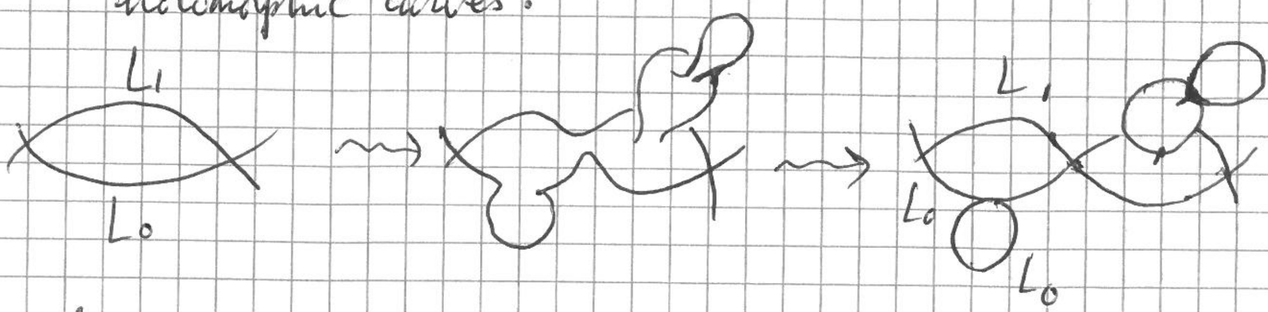
$J$-hol curve eqn $\rightsquigarrow$ $\frac{\partial u}{\partial s} + J_t(u)\frac{\partial u}{\partial t} = 0.$

(Defn: $\#M(p, q, \beta, J) := \begin{cases} \#M(\dots) & \text{if } i(\beta) = 1 \\ 0 & \text{else} \end{cases}$ )

then generic $J_t$ is regular (Floer-Hofer-Salamon).

## 1.4 Gromov compactness

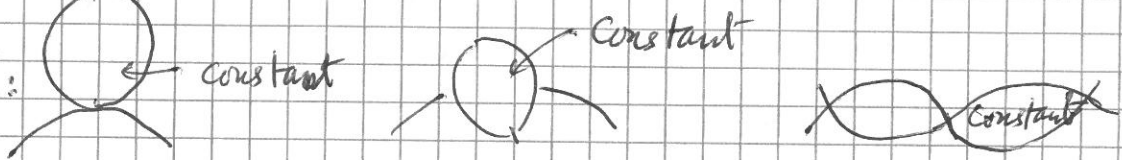
Gromov's compactness theorem says any sequence of $J$-holomorphic strips $\{u_n\}$, with $\omega(u_n) \leq E$, has a subsequence which converges (in Gromov topology) to a nodal tree of ~~non-constant~~ stable nodal holomorphic curves:



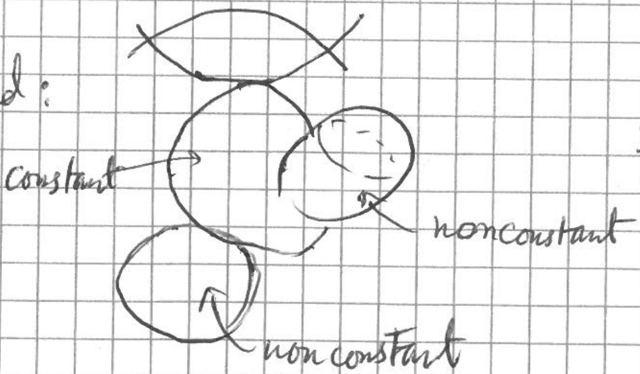

Idea: symplectic area (energy) can concentrate at
- interior points ($\Rightarrow$ sphere bubbles)
- boundary points ($\Rightarrow$ disc bubbles)
- boundary punctures ($\Rightarrow$ strip breaking).

<u>stable</u> $\leftarrow$ any constant component has discrete aut. group.

not allowed:




allowed:



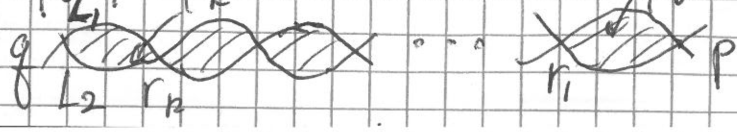

stable.

This gives us $\overline{\mathcal{M}}(p,q,\beta,J)$, the <u>Gromov compactification</u> of $\mathcal{M}(p,q,\beta,J)$.

If $\omega|_{\pi_2(M,L)}=0$ $\left(\Rightarrow \omega|_{\pi_2(M)}=0\right)$

then any disc or sphere bubble must be <u>constant</u>.

So $\overline{\mathcal{M}}(p,q,\beta,J_t) = \displaystyle\coprod_{\Sigma\beta_i=\beta} \mathcal{M}(p,r_1,\beta_0,J_t)\times\dots\times\mathcal{M}(r_k,q,\beta_k,J_t)$



Fact: $i(\beta) = i(\beta_0) + \cdots + i(\beta_k)$.

<u>Cor</u>: If $J_t$ regular, $\omega|_{\pi_2(M, L_i)} = 0$, then

- $\mathcal{M}(p, q, \beta, J_t)$ compact 0-mfld for $i(\beta) = 1$
- if $i(\beta) = 2$, then

$$\overline{\mathcal{M}}(p, q, \beta, J_t) = \mathcal{M}(p, q, \beta, J_t) \sqcup \coprod_{\substack{r \in L_0 \cap L_1 \\ \beta = \beta_0 + \beta_1 \\ i(\beta_0) = i(\beta_1) = 1}} \mathcal{M}(p, r, \beta_0, J_t) \times \mathcal{M}(r, q, \beta_1, J_t)$$


<u>Pf</u>: If $i(\beta) < 0$, $\widehat{\mathcal{M}}(r_0, r_1, \beta, J_t) = \emptyset$.

If $i(\beta) = 0$, $\widehat{\mathcal{M}}(r_0, r_1, \beta, J_t) = 0\text{-mfld} \circlearrowleft \mathbb{R}$

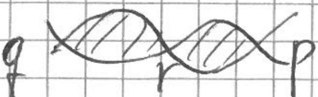
$\Rightarrow$ $\mathbb{R}$-action trivial $\Rightarrow$ $u$ const. along length

$\Rightarrow$ $u$ const ($J$-hol)

$\Rightarrow$ can't appear in $\overline{\mathcal{M}}$.

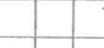
## 1.5 <u>Gluing</u>

If $J_t$ regular, $i(\beta_0) = i(\beta_1) = 1$, then a <u>gluing theorem</u> shows there is a nbhd of




$\in \overline{\mathcal{M}}(p, q, \ldots)$

homeomorphic to $[0, \delta)$

Thus, if $i(\beta) = 2$, $\overline{\mathcal{M}}(p, q, \beta, J_t)$ is a compact 1-mfld with boundary; and boundary points are .

Hence, $\#(\text{boundary points}) = 0 \pmod 2$

boundary points are in 1-1 correspondence with summands of $q$-coefficient of $\partial(\partial(p))$ $\Rightarrow$ $\partial^2 = 0$.

$\Rightarrow$ completes construction of $(CF(L_0, L_1), \partial)$, hence of $HF(L_0, L_1)$.