Sheridan lecture 2 28/06/2016

11:00 am

Webpage: web.math.princeton.edu/~nsher/jussieu.html

Text: Auroux, A beginner's introduction to Fukaya categories

Lect 1 $\subset$ [Au, §1]

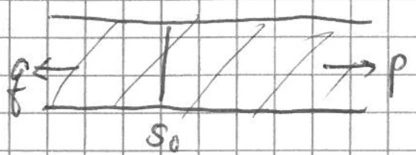
Lect 2 $\subset$ [Au, §2] $\cup$ {examples}

Lect 3 $\subset$ [Au, §4]

$\mathbb{R} \circlearrowright \hat{\mathcal{M}}(p, q, \beta, J_t)$ $\quad a \cdot u(s,t) = u(s+a, t)$

<u>Lem</u>: $a \cdot u = u \iff a = 0$ <u>or</u> $u = \text{const}$.

<u>Pf</u>:



$\lim_{k \to \infty} u(s_0 + ka, t) = p = u(s_0, t)$

<u>Lem</u>: $u_i \longrightarrow u$, $a_i \cdot u_i \longrightarrow u \iff a_i \to 0$ or $u = \text{const}$.

<u>Cor</u>: $(\hat{\mathcal{M}} \setminus \{\text{const}\}) / \mathbb{R}$ is a mfld of dim $i(\beta) - 1$.

1.6. <u>Grading</u>

<u>Defn</u>: $\mathcal{G}(n) := \{\text{linear Lag subspaces } L \subset (\mathbb{C}^n, \omega_{std})\}$

<u>Lem</u>: $\mathcal{G}(n) \cong U(n)/O(n)$

$\Rightarrow \pi_1(\mathcal{G}(n)) \cong \mathbb{Z}$

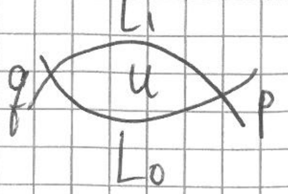
μ = <u>Maslov index</u>.

Extend to $\mu: \mathcal{P}\mathcal{G}(n) \longrightarrow \mathbb{Z}$

$\mathcal{P}\mathcal{G}(n) = \{\text{cts maps } \rho: [0,1] \longrightarrow \mathcal{G}(n) : \rho(0) \pitchfork \rho(1)\}$

μ = unique map s.t.

- cts
- $\mu(\rho_1 \times \rho_2) = \mu(\rho_1) + \mu(\rho_2)$
- $\mu(e^{i\pi N t}) = \lfloor N \rfloor + 1$ in 1-dim case. $\quad (N \notin \mathbb{Z})$



$u^*TM \cong \mathbb{R} \times [0,1] \times \mathbb{C}^n$

choose ρ_p from T_pL_0 to T_pL_1,
ρ_q from T_qL_0 to T_qL_1.

concatenate $\longrightarrow \tilde{\rho} : S^1 \to \mathcal{G}(n)$

<u>Fact</u>: $\underset{\displaystyle \dim \widehat{\mathcal{M}}(p,q,\beta,J_t)}{\overset{\shortparallel}{i(\beta)}} = \mu(\tilde{\rho}) - \mu(\rho_p) + \mu(\rho_q)$

$\boxed{\text{indep of } \rho_p, \rho_q \text{ follows from } \mu(\rho_1 \# \rho_2) = \mu(\rho_1) + \mu(\rho_2),\ \rho_1 \in P\mathcal{G},\ \rho_2 \in \pi_1 \mathcal{G}}$

Consider $\mathcal{G}(n) \hookrightarrow \mathcal{G}M \leftarrow$ Lag. subspaces of TM.

$$\mathcal{G}M \to M$$

Let $\widetilde{\mathcal{G}(n)} \hookrightarrow \widetilde{\mathcal{G}M}$ be a fiberwise univ. cover of $\mathcal{G}M$

$$\widetilde{\mathcal{G}M} \to M$$

(exists $\iff 2c_1(TM) = 0$)

(need not be unique)

<u>Defn</u>: A <u>grading</u> of $L \subset M$ is a lift.

$$\begin{tikzcd} & \widetilde{\mathcal{G}M} \arrow[d] \\ L \arrow[ur, dashed] \arrow[r] & \mathcal{G}M \end{tikzcd}$$

(exists if $\pi_1(L) \longrightarrow \pi_1(\mathcal{G}M) \longrightarrow \mathbb{Z}$ vanishes)

classifies $\widetilde{\mathcal{G}M}$

$\mu_L \in H^1(L;\mathbb{Z})$

Maslov class of L (does not dep on choices of covers.)

Suppose L_0, L_1 are equipped with gradings.

$\forall\, p \in L_0 \cap L_1$ ($L_0 \pitchfork L_1$)

$\exists!$ htpy class of path $\rho_p : T_pL_0 \to T_pL_1$ that lifts to a path from

$\widetilde{T_pL_0} \to \widetilde{T_pL_1}$

<u>Defn</u>: $|p| := \mu(\rho_p) \Rightarrow \mathbb{Z}$-grading on $CF(L_0, L_1) = \Lambda\langle L_0 \cap L_1\rangle$

<u>Lem</u>: $|\partial(p)| = |p| + 1$.

<u>Pf</u>: $i(\beta) = \mu(\tilde{\rho}) - \mu(\rho_p) + \mu(\rho_q)$ Recall: $i(\beta) = 1$.

$\mu(\tilde{\rho}) = 0$ (lifts to $\widetilde{G(n)}$)

($|p|$ depends on choices of covers $\widetilde{GM}$, ρ_p)

<u>Rmk</u>: If L_i are <u>spin</u>, can <u>orient</u> $\mathcal{M}(p, q, \beta, J_t)$

$\rightsquigarrow$ can count with signs

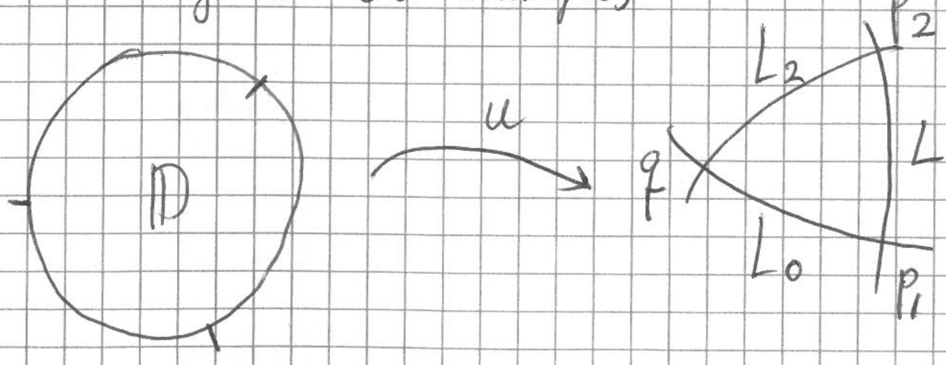
$\rightsquigarrow$ work over $\Lambda_{\mathbb{K}}$ for char $(\mathbb{K}) \neq 2$.

(end of lecture 1)

2. <u>Product structures</u>

2.1 <u>Product</u>

Counting J-hol. triangles



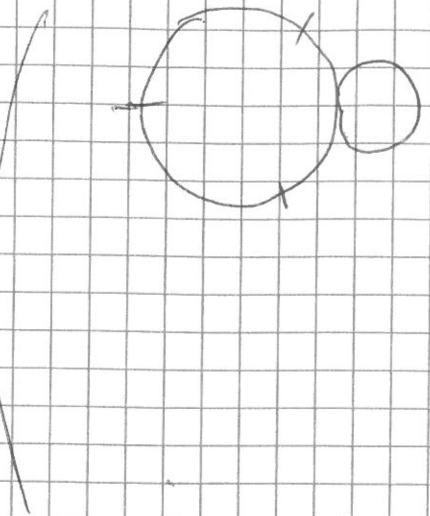
$\bar\partial_J u = 0 \iff Du \circ j = J_z(u) \circ Du$

j: cpx struc on $\mathbb{D}$.

As before, have moduli space $\mathcal{M}(p_1, p_2, q, \beta, J_z)$ of such maps, with $[u] = \beta$, which is a mfld of dim $= i(\beta)$ for generic J_z.

<u>Defn</u>: $m_2 : CF(L_1, L_2) \otimes_\Lambda CF(L_0, L_1) \to CF(L_0, L_2)$

$$m_2(p_2, p_1) := \sum_{q, \beta} \# \mathcal{M}(p_1, p_2, q, \beta, J_z)\, \underset{\underset{\Lambda}{\cap}}{T^{\omega(\beta)}}\, q$$



(If a hol. disc bubbles off, then J is domain-indep, cannot achieve transversality for multiply covered discs. This is ruled out $\omega_2|_{\pi_2(M,L)}=0$. Above can be done in classical techniques.)

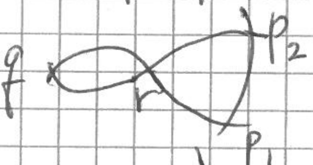
If L i graded, $i(\beta)=|q|-|p_1|-|p_2|$

$$\Rightarrow |m_2(p_2,p_1)|=|p_1|+|p_2|$$

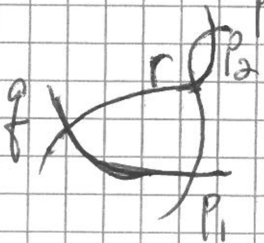
What relations m_2 satisfies?

1-dim'l component of $\overline{\mathcal{M}}(p_1,p_2,q,\beta,J_Z)$ is

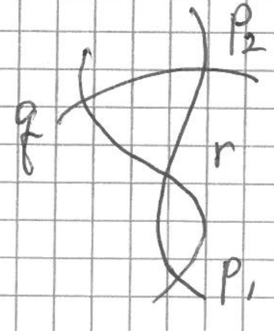
$\mathcal{M}(p_1,p_2,q,\beta,J_Z)\ \sqcup$



$\sqcup$



$\sqcup$



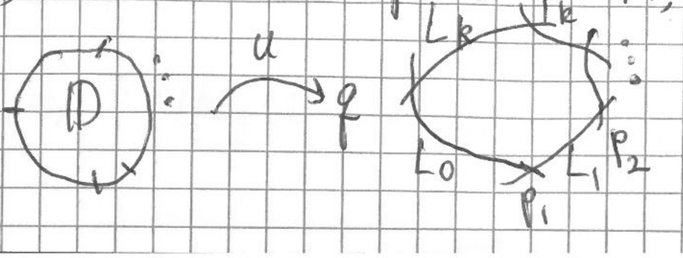
$(\omega|_{\pi_2(M,L)}=0)$.

$$\Rightarrow \partial m_2(p_2,p_1)+m_2(\partial p_2,p_1)+m_2(p_2,\partial p_1)=0$$

$$\Rightarrow \text{get map } HF(L_1,L_2)\otimes HF(L_0,L_1)\xrightarrow{m_2} HF(L_0,L_2).$$

2.2 A_∞ products

More generally, consider moduli spaces $\mathcal{M}(p_1,\dots,p_k,q,\beta,J_Z)$ of hol. maps



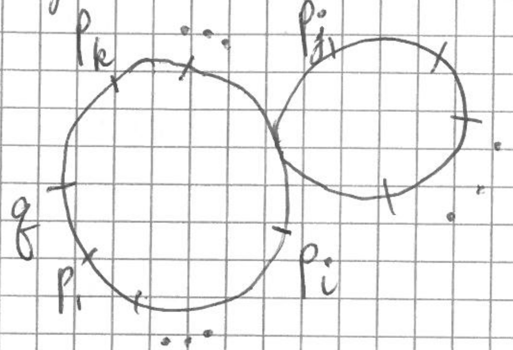
(requires cyclic symmetry (hence that of J_z) if doing $k+1$ inputs, 0 output, hard to achieve transversality)

Counting 0-dim'l component of mod. space of such J-hol. maps defines a map

$m_k: CF(L_{k-1}, L_k) \otimes \dots \otimes CF(L_0, L_1) \to CF(L_0, L_k)$

(of deg $2-k$ if L_i graded)

Counting boundary of 1-dim'l compactified moduli space gives relation:



$\Rightarrow \sum m^{k-j+i}(p_k, \dots, m^{j-i}(p_j, \dots), p_i, \dots, p_1) = 0$

$\Rightarrow$ We have partially defined an A_∞ category.

Would like to define $Fuk(M, \omega)$:

- Obj are $L \subset M$ Lagrangian, $\omega|_{\pi_2(M,L)} = 0$ (graded, spin)
- $Hom(L_0, L_1) = CF(L_0, L_1)$.
- A_∞ structure maps m_k.

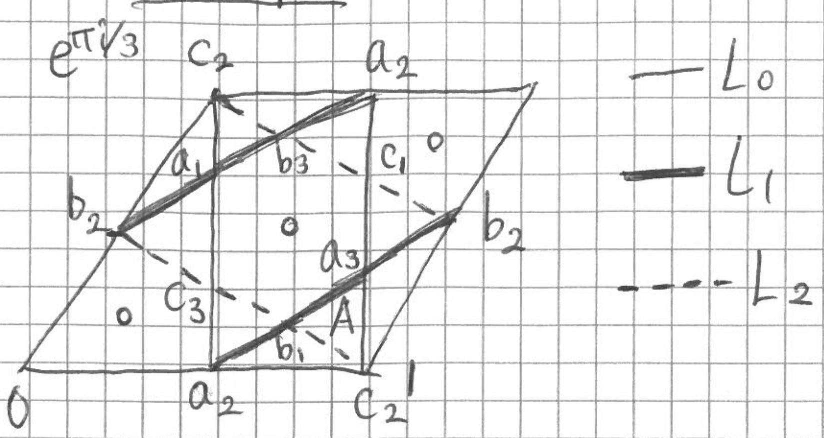
Issue $Hom(L_0, L_1)$ only defined for $L_0 \pitchfork L_1$

$\longrightarrow$ choose system of Ham. perturbations to make $L_0 \pitchfork \psi_{0,1}(L_1)$

$Hom(L_0, L_1) := \Lambda \langle L_0 \cap \psi_{0,1}(L_1) \rangle$ [Seidel's book].

$h(Fuk(M, \omega))$ is (deg-0 part of) Donaldson-Fukaya category.

2.3 Example



$\mathcal{A}$ = full subcat. of $h(\mathrm{Fuk}(T^2 \setminus 3\text{pts}))$

Ex: For appropriate gradings of L_i,
(rel. $\widetilde{GM} \to GM$ extends to T^2).

$HF^\circ(L_0, L_1) = \Lambda\langle a_1, a_2, a_3\rangle$, $|a_i| = 0$
$HF^\circ(L_1, L_2) = \Lambda\langle b_1, b_2, b_3\rangle$, $|b_i| = 0$
$HF^\circ(L_0, L_2) = \Lambda\langle c_1, c_2, c_3\rangle$, $|c_i| = 0$

$\Rightarrow$ morph. spaces in other direction are spanned by $a_i^\vee$ in deg = 1.

$HF^\circ(L_i, L_i) \cong H^\circ(S^1) = \Lambda\langle e, \theta\rangle$, $|e| = 0$, $|\theta| = 1$.

Ex: $m_2(b_j, a_i) = \begin{cases} \pm c_k & \text{when } \{i,j,k\} = \{1,2,3\} \\ 0 & \text{if } i = j \end{cases}$

Rmk: $h(\mathrm{Fuk}(M,\omega))$ has identity elements
$e \in H^0(L) \cong HF^0(L, L)$.

$$\begin{tikzcd} L_0 \arrow[r, bend left, "a_1"] \arrow[r, "a_2"] \arrow[r, bend right, "a_3"'] & L_1 \arrow[r, bend left, "b_1"] \arrow[r, "b_2"] \arrow[r, bend right, "b_3"'] & L_2 \end{tikzcd}$$

relations:
$b_i a_j = b_j a_i \ (= c_k)$ for $i \neq j$
$b_i a_i = 0$.

Compare with $\mathrm{Coh}(\mathbb{P}^2)$
(hint of mirror symmetry) $\mathcal{O}, \Omega^1(1), \Omega^2(2)$.