30/06/2016

Sheridan 3

3 Triangulated structure

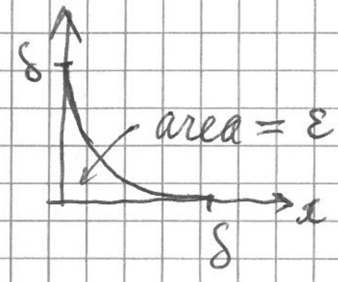
Recall: (M,ω) - symp. mfd $\longrightarrow Fuk(M,\omega)$ - A_∞ cat.

Obj = Lagrangian $L \subset M$, $\omega|_{\pi_2(M,L)} = 0$.

$\rightsquigarrow Tria(Fuk(M,\omega))$.

3.1 Lag. connect. sum & mapping cones

Let $f_\varepsilon : (0,\delta] \longrightarrow [0,\delta]$ have graph:



$$L_\varepsilon := \left\{ p_i = \frac{\partial}{\partial q_i} f_\varepsilon(\|q\|) \right\} \subset (\mathbb{R}^n \times \mathbb{R}^n, \omega_{std})$$

Lagrangian

$L_\varepsilon \setminus B_\delta(0) \cong (\mathbb{R}^n \times 0 \cup 0 \times \mathbb{R}^n) \setminus B_\delta(0)$

Let $L_1, L_2 \subset M$ be Lag., $p \in L_1 \cap L_2$ transverse intersection point. Identify nbhd of p with $B_\delta(0)$ s.t. $L_1 \longmapsto \mathbb{R}^n \times 0$, $L_2 \longmapsto 0 \times \mathbb{R}^n$.

Glue in a copy of $L_\varepsilon \longrightarrow$ get $L_1 \#_\varepsilon L_2$.

If $L_1 \cap L_2 = \{p\}$, $L_1 \#_\varepsilon L_2$ embedded.

$\Rightarrow$ new object of $Fuk(M,\omega)$.

ind of f_ε up to Ham. isot., also ind of ε.

Claim: In $Tria(Fuk(M,\omega))$,

$$L_1 \#_\varepsilon L_2 \cong C(L_2 \xrightarrow{T^{-\varepsilon} p} L_1)$$

cone

Follows from (unpublished) work of FOOO.

E.g. Let L_3 be another Lag. Assume $L_3 \cap L_1 \cap L_2 = \emptyset$.

Then for $\varepsilon > 0$ small, $L_3 \cap (L_1 \#_\varepsilon L_2) = (L_3 \cap L_1) \cup (L_3 \cup L_2)$

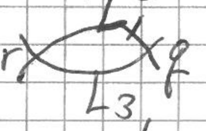
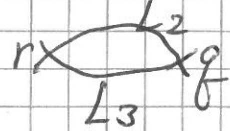
$\Rightarrow \mathrm{Hom}(L_3, L_1 \#_\varepsilon L_2) \cong \mathrm{Hom}(L_3, L_1) \oplus \mathrm{Hom}(L_3, L_2)$.

Claim: $(\mathrm{Hom}(L_3, L_1 \#_\varepsilon L_2), m_1) \cong (\mathrm{Hom}_{Tw(Fuk)}(L_3, L_2 \xrightarrow{T^{-\varepsilon} p} L_1), m_1^{Tw(Fuk)})$

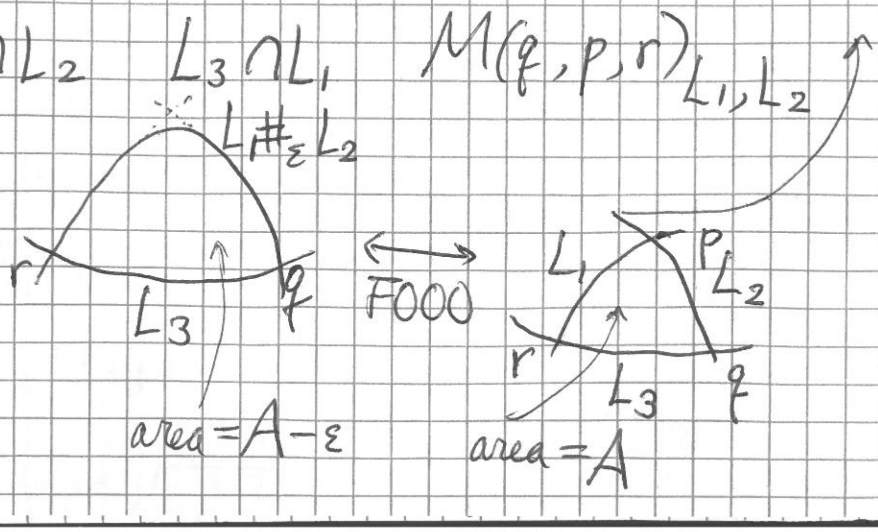
iso, not just quasi-iso

Let $q, r \in L_3 \cap (L_1 \#_\varepsilon L_2)$. FOOO's result says (roughly): the space of J-hol. discs w/ bdry on $L_1 \#_\varepsilon L_2$ is isom. to the space of J-hol. discs w/ bdry on L_1 and L_2, which are allowed to 'switch' from L_2 to L_1 at p (but not back from L_1 to L_2).

For <u>strips</u>:

$q \in$	$r \in$	$\mathcal{M}(q,r)_{L_1 \#_\varepsilon L_2}$
$L_3 \cap L_1$	$L_3 \cap L_1$	$\mathcal{M}(q,r)_{L_1}$ 
$L_3 \cap L_2$	$L_3 \cap L_2$	$\mathcal{M}(q,r)_{L_2}$ 
$L_3 \cap L_1$	$L_3 \cap L_2$	$\emptyset$
$L_3 \cap L_2$	$L_3 \cap L_1$	$\mathcal{M}(q,p,r)_{L_1, L_2}$

Pic:



$\Rightarrow m_1$ is given by

$$\begin{tikzcd} \mathrm{Hom}(L_3, L_2) \arrow[loop below, "m_1"] \arrow[rr, bend right, "m_2(T^{-\varepsilon}p, -)"'] & \oplus & \mathrm{Hom}(L_3, L_1) \arrow[loop below, "m_1"] \end{tikzcd}$$

(and A_∞ products $m_k^{Tw(Fuk)}$ work similarly).

$$m_k^{Tw(Fuk)} = \sum_{\substack{i_0 + \dots + i_k \\ = t}} m_{k+t}(\delta^{i_0}, -, \delta^{i_1}, -, \dots, \delta^{i_k})$$

δ MC element.

3.2 <u>(split-) generation</u>

<u>Defn:</u> The objects $L_1, \dots, L_k$ <u>generate</u> an A_∞ cat $\mathcal{A}$ if all obj. of $\mathcal{A}$ are isomorphic in $Tria(\mathcal{A})$ to a twisted complex built from L_i.

They <u>split-generate</u> if every obj. of $\mathcal{A}$ is isom. in $Tria(\mathcal{A})$ to a direct summand of such.

E.g. $M = T^2 = \mathbb{R}^2/\mathbb{Z}^2$

$L_1 = \{x = 0\}$ $\quad L_2 = \{y = 0\}$.

<u>Ex:</u> You can get curves in any homology class on T^2 by taking iterated Lag. connect. sums of L_1 and L_2.

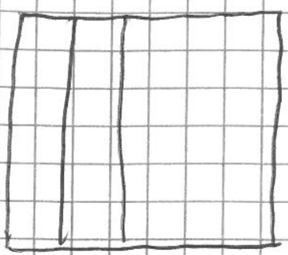
But do not generate $Fuk(M)$.

Let $\theta \in \Omega^1(T^2 \setminus \{\frac{1}{2}, \frac{1}{2}\})$, $d\theta = \omega$, $\theta|_{L_1} = \theta|_{L_2} = 0$.

For any $(\bigoplus_i L_i[n_i], \delta_{ij}) \in Tw(Fuk)$

set $G(\bigoplus_i L_i[n_i], \delta_{ij}) := \sum_i (-1)^{n_i} \int_{L_i} \theta \in \mathbb{R}/\mathbb{Z}$

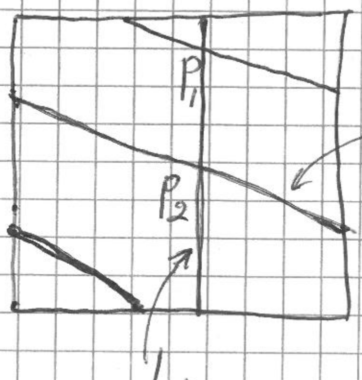
Prop (Abouzaid) If $\mathcal{L}$, $\mathcal{K}$ are isomorphic tw. cpx.,
then $G(\mathcal{L}) = G(\mathcal{K})$
Further, $G(\mathrm{Cone}(\mathcal{L} \xrightarrow{f} \mathcal{K})) = G(\mathcal{K}) - G(\mathcal{L})$



same homology class, not iso Lags.
S^1 family, G is a complete invariant.

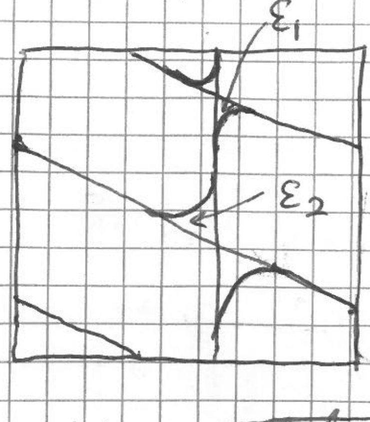
Cor: L_1 & L_2 generate the subcat. of $Fuk(M)$ consisting of balanced curves (those with $\int_L \theta = 0$).

Claim: L_1 & L_2 split-generate $Fuk(M)$.

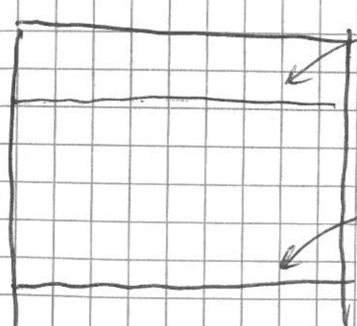


$L_3 = \{y = -\frac{1}{2}x\}$

$\mathrm{Cone}(L_3 \xrightarrow[T^{-\varepsilon_1} P_1 + T^{-\varepsilon_2} P_2]{} L_1)$



Ham. isot.
$\sim$

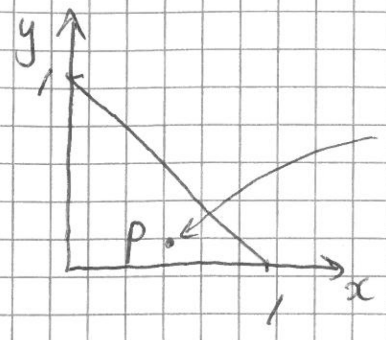


$\{y = \frac{1}{4} + \varepsilon_2 - \varepsilon_1\}$

$\{y = -\frac{1}{4} + \varepsilon_1 - \varepsilon_2\}$

split-generated by L_1 & L_2
tuning $\varepsilon_2, \varepsilon_1$ can split-gen. curves with any value of $\int_L \theta$ in this homology class.

3.3 Example: $\mathbb{C}P^2$



$\pi : \mathbb{C}P^2 \longrightarrow \Delta$

(p) Lag. torus fibre

$L_p := \pi^{-1}(p)$

Claims:

① $\mathrm{Fuk}(\mathbb{C}P^2) = \coprod_{\lambda \in \Lambda} \mathrm{Fuk}(\mathbb{C}P^2)_\lambda$

(because $\omega(\pi_2(M)) \neq 0$) $\Rightarrow \omega(\pi_2(M, L)) \neq 0$

see details for generalization to this setting in Sheridan's online notes.

② $L_p \in \mathrm{Fuk}(\mathbb{C}\mathbb{P}^2)_{\lambda(p)}$ $\quad \lambda(p_1, p_2) = T^{p_1} + T^{p_2} + T^{1-p_1-p_2}$

sum of affine distances of p to three sides of triangle.

$L_p \cong 0$ unless $p = (\frac{1}{3}, \frac{1}{3})$.

③ Thm (Abouzaid - FOOO) (unpublished)

$L_{(\frac{1}{3}, \frac{1}{3})}$ split generates $\mathrm{Fuk}(\mathbb{C}\mathbb{P}^2)_{3T^{\frac{1}{3}}}$,

if we work over $\Lambda_{\mathbb{C}}$ (which we can work over because T^2 is spin).

Note: $\mu_L \neq 0$ for $L = L_p \rightsquigarrow$ Fuk <u>not</u> $\mathbb{Z}$-graded,

but does have a $\mathbb{Z}/2$-grading.

④ $HF(L_{(\frac{1}{3}, \frac{1}{3})}, L_{(\frac{1}{3}, \frac{1}{3})}) \cong Cl_2(\Lambda_{\mathbb{C}}) \cong \mathrm{Mat}_{2\times 2}(\Lambda_{\mathbb{C}})$

$\Rightarrow$ L has "formal direct summand" K with

$\mathrm{Hom}(K, K) \cong \Lambda_{\mathbb{C}}$ and K also split-generates $\mathrm{Fuk}(\mathbb{C}\mathbb{P}^2)_{3T^{\frac{1}{3}}}$

idempotent completion

$\Rightarrow D^{\pi}\mathrm{Fuk}(\mathbb{C}\mathbb{P}^2)_{3T^{\frac{1}{3}}} \cong D^b(\Lambda_{\mathbb{C}})$

cat. of graded $\Lambda_{\mathbb{C}}$-vector spaces.