Sheridan N.

Summer school 2016 of the IMJ-PRG
Symplectic topology, sheaves, mirror symmetry

Introduction to the Fukaya category

Course 1: 27/06/16.

I. Lagrangian Floer cohomology

(1) Arnol'd conjecture

Webpage web.math.princeton.edu/~nsher/jussieu.html (personal page of N. Sheridan)

(M,ω) compact symplectic manifold

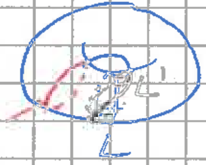
$L \subset M$ Lagrangian submanifold

$\psi \in Ham(M,\omega)$ (i.e. $\psi = \psi_1$ time-1-flow of X_t where $\omega(\cdot, X_t) = dH_t$).

<u>Def</u>: L' is Ham isotopic to L if $\exists \psi \in Ham(M)$ s.t. $L' = \psi(L)$

<u>Thm (Floer)</u> If $L' \pitchfork L$, L' Ham isotopic to L and $\omega|_{\pi_2(M,L)} = 0$, then $|L \cap L'| \geq \sum \dim H^i(L;\mathbb{Z}_2)$

<u>Example</u>:



$\phi(L)$
$\phi \in Symp(M)$
$\phi \notin Ham(M,\omega)$

Area 1 = Area 2 → at least 2 intersection points between L and L'
because flux diffeo

Here $K \cap \psi(K) = \emptyset$ is possible because K bounds a disk $(\omega|_{\pi_2(M,K)} \neq 0)$
with nonzero area

<u>Sketch of proof</u> Floer associated to L_0, L_1 (pair of transverse Lagrangian submanifolds with $\omega|_{\pi_2(M;L_i)} = 0$) a cochain complex $(CF(L_0,L_1), \partial)$ with cohomology $HF(L_0,L_1)$ such that:

(Ham-isotopic invariance) 1) L_i' Ham-isotopic to L_i $(i=0,1) \Rightarrow HF(L_0,L_1) \cong HF(L_0',L_1')$

2) $CF(L_0,L_1)$ is a Λ-vector space with basis in bijection with $L_0 \cap L_1$ (field extension of $\mathbb{Z}_2$)

3) $HF^i(L,L) \simeq H^i(L;\Lambda)$

Sketch of proof 1)–3) suffices to prove Floer's theorem

② The definition

$\mathbb{K}$ is a field.

Def: Novikov field over $\mathbb{K}$: $\Lambda_{\mathbb{K}} := \left\{ \sum_{i=0}^{\infty} a_i T^{\lambda_i} ; a_i \in \mathbb{K}, \lambda_i \in \mathbb{R}, \lim_{i\to\infty} \lambda_i = +\infty \right\}$

(Recall: Laurent series over $\mathbb{K} = \left\{ \sum_{i=-\infty}^{\infty} a_i T^{i} ; a_i \in \mathbb{K} \right\}$)

$$\Lambda = \Lambda_{\mathbb{Z}_2}$$

Define $CF(L_0, L_1) := \Lambda \langle L_0 \cap L_1 \rangle$

If $p \in L_0 \cap L_1$, $\partial p = \sum_{\substack{q \in L_0 \cap L_1 \\ B}} \underbrace{\# \mathcal{M}(p, q, B, J) \cdot T^{\omega(B)}}_{\in \Lambda} q$

$\mathcal{M}(p, q, B, J)$ depends on J = an ω-compatible almost complex structure: $J \in \mathrm{End}(TM)$, $J^2 = -\mathrm{Id}$

$\omega(\cdot, J \cdot)$ = Riemannian metric

$\tilde{\mathcal{M}}(p, q, B, J)$ is the set of J-holomorphic strips, these are smooth maps $u : \underset{(s,\, t)}{\mathbb{R} \times [0,1]} \longrightarrow M$ such that

$\frac{\partial u}{\partial s} + J(u) \frac{\partial u}{\partial t} = 0$ (J-hol. curve equation)

$u(s, 0) \in L_0$, $u(s, 1) \in L_1$ and $\lim_{s \to +\infty} u(s,t) = p$, $\lim_{s \to -\infty} u(s,t) = q$

$[u] = B$

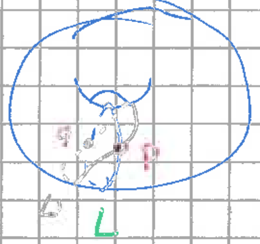
. $\omega(B) = \int_{\mathbb{R} \times [0,1]} u^* \omega \underset{\text{exercise}}{=} \iint \left|\frac{\partial u}{\partial s}\right|^2 ds\, dt \geqslant 0$ with equality iff u is a constant

. $\mathbb{R}$ acts on $\tilde{\mathcal{M}}(p, q, B, J)$ by translation: $a \in \mathbb{R}$, $u \in \tilde{\mathcal{M}}$ then $(a \cdot u)(s, t) = u(s + a, t)$ $\quad a \cdot u \in \tilde{\mathcal{M}}$.

Therefore we define $\mathcal{M}(p, q, B, J) = \tilde{\mathcal{M}}(p, q, B, J) / \mathbb{R}$

picture of a holomorphic strip u

Example



$$CF(L,L') = \Lambda\langle p,q\rangle$$
$$\partial p = \left(T^{\text{Area}_1} + T^{\text{Area}_2}\right) q$$

If $\text{Area}_1 = \text{Area}_2$ (which is the case when L' is Ham-isotopic to L)

then $T^{A_1} + T^{A_2} = 0 \Rightarrow \partial = 0 \Rightarrow HF(L,L') \cong \Lambda\langle p,q\rangle$

If $A_1 \neq A_2$ (WLOG $A_1 < A_2$), then $(T^{A_1}+T^{A_2})^{-1} = T^{-A_1}(1+T^{A_2-A_1}+T^{2(A_2-A_1)}+\dots$

$\Rightarrow HF(L,L') \cong 0$

We can expect this result by the Ham-isotopic invariance because if $A_1 \neq A_2$ then L' is ham-isotopic to L'' such that $L'' \cap L = \emptyset$

3 Transversality

How come we can count holomorphic strips?

The defining PDE of $\hat{\mathcal{M}}$ is a Fredholm problem.

$\Rightarrow$ When J is regular, $\hat{\mathcal{M}}(p,q,B,J)$ has the structure of an $i(B)$-dimensional ~~problem~~ manifold (where $i(B)$ is the index of the Fredholm problem).

We would like that J is "generically" regular. To achieve this, we need to allow J to vary in the t-direction

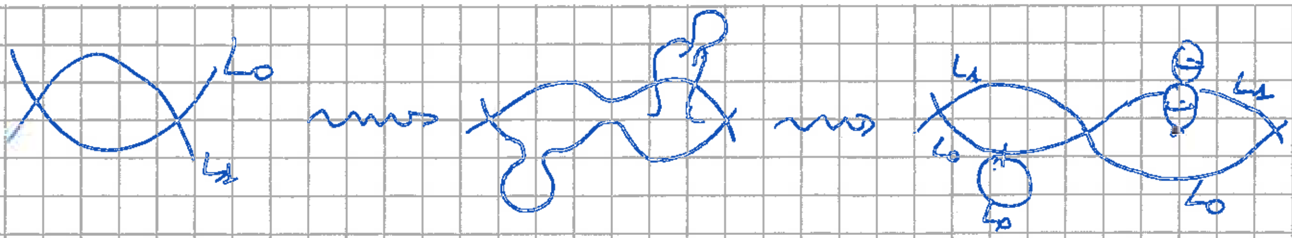
J-hol eqn $\rightsquigarrow \dfrac{\partial u}{\partial s} + J_t(u)\dfrac{\partial u}{\partial t} = 0$

(Remark $\# \mathcal{M}(q,p,B,J) = \begin{cases} \# \mathcal{M}(\ ) & \text{if } i(B) = 1 \\ 0 & \text{else} \end{cases}$)

Then generic J_t is regular (Floer - Hofer - Salamon)

4 Compactness

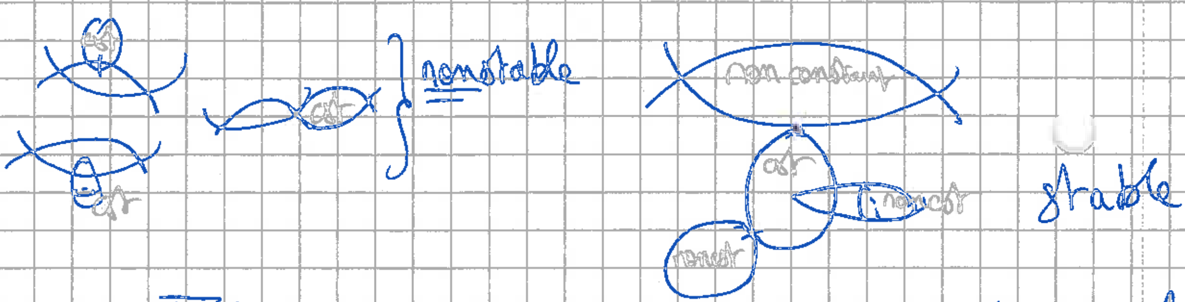
Gromov's compactness theorem says that any sequence of J-holomorphic strips (u_n) with $\omega(u_n) < E$ has a subsequence which converges (in Gromov-topology) to a nodal tree of (~~nonconstant~~ stable) nodal holomorphic curves



Idea: Symplectic area ('energy') can concentrate at
- interior points ($\Rightarrow$ sphere bundles)
- boundary points ($\Rightarrow$ disc bubbles)
- boundary punctures ($\Rightarrow$ strip breaking)

"Def" of stable: any constant component has discrete aut group

Example: of stable and non stable nodal holomorphic curves

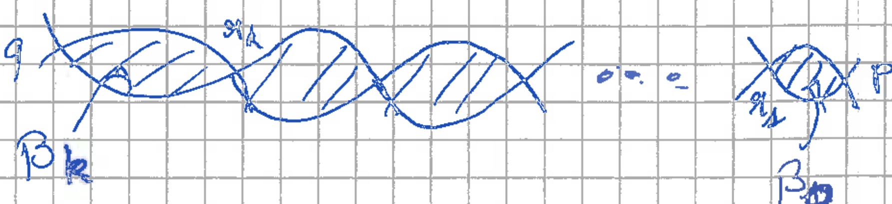


This gives us $\overline{\mathcal{M}}(p,q,\beta,J)$ the Gromov compactification of $\mathcal{M}(p,q,\beta,J)$

If $\omega|_{\pi_2(M,L)}=0$ $(\Rightarrow \omega|_{\pi_2(M)}=0)$ then any disc or sphere bubbles must be constant.

$\Rightarrow$ there are no disc or sphere bubbles in stable nodal curves and

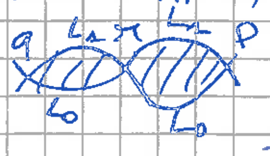
$$\overline{\mathcal{M}}(p,q,\beta,J_t) := \bigsqcup_{\sum \beta_i=\beta} \mathcal{M}(p,x_1,\beta_0,J)\times \cdots \times \mathcal{M}(x_k,q,\beta_k,J)$$



Fact: $i(\beta)=i(\beta_0)+\dots+i(\beta_k)$

Cor: If J_t is regular, $\omega|_{\pi_2(M,L_i)}=0$, then
- $\mathcal{M}(p,q,\beta,J_t)$ compact 0-mfd for $i(\beta)=1$
- If $i(\beta)=2$, then $\overline{\mathcal{M}}(p,q,\beta,J_t)=\mathcal{M}(p,q,\beta,J_t)$

$$\cup \bigsqcup_{\substack{x\in L_0\cap L_1\\ \beta=\beta_0+\beta_1\\ i(\beta_0)=i(\beta_1)=1}} \mathcal{M}(p,x,\beta_0,J_t)\times\mathcal{M}(x,q,\beta_1,J_t)$$



Pf If $i(B)<0$, $\mathcal{M}(x_0,x_1,B,J_t)=\emptyset$.

If $i(B)=0$, $\mathcal{M}(x_0,x_1,B,J_t)=$ 0-mfd $\circlearrowleft \mathbb{R}$.

$\Longrightarrow$ $\mathbb{R}$-action trivial $\Rightarrow$ u constant along length (s-dir)

$\Rightarrow$ u constant (J-hol eqn)

$\Rightarrow$ can't appear in $\overline{\mathcal{M}}$.

5 Gluing

If J_t regular, $i(B_0)=i(B_1)=1$ then a gluing theorem shows there's a neighborhood of  $\in \overline{\mathcal{M}}$ homeomorphic to $[0,\delta[$.

Thus if $i(B)=2$, $\overline{\mathcal{M}}(p,q,B,J_t)$ is a compact 1-manifold and boundary points are 

Hence, $\#$(boundary points) $=0 \pmod 2$. Boundary points are in 1-1 correspondence of summands of $\partial(\partial p)$ $\Rightarrow \partial^2=0$

$\Rightarrow$ completes construction of $(CF(L_0,L_1),\partial)$ hence of $HF(L_0,L_1)$

y danskiy M.

Summer school 2016 of the IMJ-PRG
Symplectic topology, sheaves, mirror symmetry

Complementary lecture : Introduction to the Fukaya category.

Course 1 : 27/06/16

Some topics

#1 Floer cohomology as Morse theory on path space

#2 Floer cohomology of 0-section in T^*N vs Morse cohomology on N.

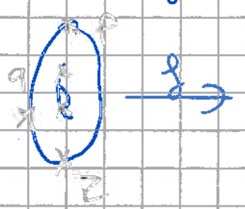
#3 Example of Gromov bubbling of closed holom. sphere

#4 Example computation of HF in $M = \mathbb{R}^2$ (exact Lagrangians)

I. Morse theory

M smooth mfd. We say that $f : M \to \mathbb{R}$ is Morse if for all $p \in \mathrm{Crit}(f)$ (i.e. $df_p = 0$), $\mathrm{Hess}_p(f)$ is non degenerate

Example:



(f = heigth function) has 4 critical points

Pick a metric g on M and suppose (g, f) is Palais-Smale (genericity condition)

To this, associate Morse-Witten complex

$CM_*(M, f, g) = \bigoplus_{p \in \mathrm{Crit}(f)} \mathbb{Z}_2 \langle p \rangle$ (we choose here to work in the coefficient field $\mathbb{K} = \mathbb{Z}_2$)

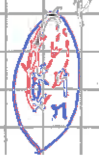
Differential: The coefficient of q in ∂p is the number of solutions $\gamma : \mathbb{R} \to M$ to

- $\dot{\gamma} = (\nabla_g f)(\gamma)$
- $\lim_{t \to +\infty} \gamma(t) = q$, $\lim_{t \to -\infty} \gamma(t) = p$

We note $\mathcal{M}(p, q)$ the set of such γ

Graduation: $\deg(p) = \#$ negative eigenvalues of $\mathrm{Hess}_p f$

Example:



$\partial p = (q + q) + (r + r)$

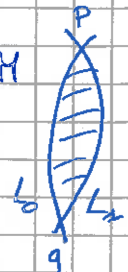
$\deg(p) = 2$
$\deg(q) = \deg(r) = 1$
$\deg(z) = 0$.

④

Prop: $\mathcal{M}(p,q)$ is a manifold of dimension $\deg p - \deg q$
$\mathbb{R}$ acts properly (by translation) on $\mathcal{M}(p,q)$ and $\mathcal{M}/_{\mathbb{R}}$ is a mfd of dim $\deg p - \deg q - 1$.

① Floer cohomology as Morse theory

(M,ω) symp mfd. In Floer cohomology, we count strips $u : \mathbb{R}\times[0,1] \longrightarrow M$
$u(\cdot,0)\in L_0$
$u(\cdot,1)\in L_1$



Idea: Strip $u(s,t)$ is a trajectory of paths $\big(u(s,\cdot)\ [0,1] \longrightarrow M\big)_{s\in\mathbb{R}}$ with endpoint conditions.

Def: Path space $\mathcal{P} = \{\gamma : [0,1] \xrightarrow{\ \mathcal{C}^\infty\ } M$ with $\gamma(0)\in L_0,\ \gamma(1)\in L_1\}$
(∞-dimensional)

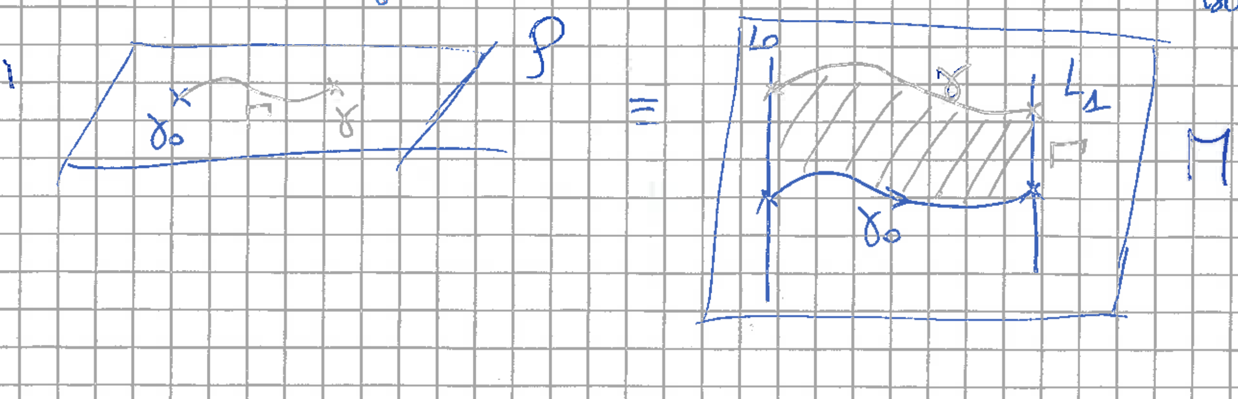
Remark: If we wanted to do something precise, we would not use $\mathcal{C}^\infty$ but other regularity condition to get a Banach manifold
But this only works as an inspiration and we would run into analytic problems if we tried to formalize this formulation
These problems come from the bubbling phenomenon and that's why we do Floer theory and not Morse theory on the path of space

Then $T_\gamma\mathcal{P} = \{\eta\in\Gamma(\gamma^*TM) \mid \eta(0)\in T_{\gamma(0)}L_0,\ \eta(1)\in T_{\gamma(1)}L_1\}$

"Morse function" is the action defined on $\tilde{\mathcal{P}}$ and not $\mathcal{P}$

$\boxed{\mathcal{A}(\gamma,\Gamma) = -\int_\Gamma \omega}$ where $\tilde{\mathcal{P}}$ is the universal cover

$\tilde{\mathcal{P}} = \{(\gamma,\Gamma) \mid \Gamma$ is a isotopy of γ to a fixed γ_0 path in the same isotopy class$\}$



"Metric" on path space: Choose g metric compatible with ω on M and define for $\eta, \xi \in T_\gamma P$, $G(\eta,\xi) = \int_0^1 g(\eta(t),\xi(t))\,dt$

Critical points of $\mathcal{A}$: consider $\eta \in T_\gamma \mathcal{P}$

$d\mathcal{A}_\gamma(\eta) = ?$

For $\varepsilon > 0$, take $\gamma_\varepsilon = \exp_{\gamma(t)}(\varepsilon \eta(t))$

$$\frac{\mathcal{A}(\gamma_\varepsilon) - \mathcal{A}(\gamma)}{\varepsilon} = \int_{\text{Strip}} \frac{\omega}{\varepsilon}$$

Finally we get $d\mathcal{A}_\gamma(\eta) = \int_0 \omega(\dot\gamma(t), \eta(t))$

Critical points are points where $d\mathcal{A}_\gamma(\eta)$ for all η The only way is $\dot\gamma = 0$ i.e. $\gamma = \text{constant}$

$\to$ Critical points are in bijection with intersection points $L_0 \cap L_1$

Gradient flowlines?

g is compatible with ω, take J s.t. $g(\cdot,\cdot) = \omega(\cdot, J\cdot)$ (or $\omega(\cdot,\cdot) = g(J\cdot,\cdot)$)

$$d\mathcal{A}_\gamma(\eta) = \int_0^1 \omega(\dot\gamma(t),\eta(t))\,dt = \int_0^1 g(J\dot\gamma, \eta)\,dt = G(J\dot\gamma, \eta)$$

And $\nabla \mathcal{A}_\gamma = J\dot\gamma$

A gradient flowline is a $u: \mathbb{R} \to \tilde{\mathcal{P}}$ s.t. $\frac{\partial u}{\partial s} = \nabla \mathcal{A}_\gamma(u)$

$u: \mathbb{R}\times[0,1] \to M$ $\quad$ $u(s,t)$ flowline $\Leftrightarrow \frac{\partial u}{\partial s} = J\frac{\partial u}{\partial t}$

(s,t)

~~Floer side~~ ~~Morse side~~

(2) Floer cohomology of 0-section vs Morse cohomology on N

Floer side	Morse side
T^*N, 0-section $= L_0$	N
$L_1 = \text{graph}(df) \subset T^*N$	$f: N \to \mathbb{R}$
$df = 0 \Rightarrow L_1 \cap L_0 =$ Crit pts of f	Crit points of f
$L_0 \pitchfork L_1$	f Morse i.e. $\text{Hess}_p f$ nondeg

Next: Holomorphic strips $u : \mathbb{R}\times[0,1] \longrightarrow T^*N$
$u(s,0) \in L_0$
$u(s,1) \in L_1$.

Claim: $H = f\circ\pi : T^*N \longrightarrow \mathbb{R}$ hamiltonian on T^*N.
$\varphi_1^H(L_0) = L_1$. $\tilde{u}(s,t) = \varphi_t^{-1}(u(s,t))$

u verifies $\frac{\partial u}{\partial s} = J\frac{\partial u}{\partial t}$ and $\tilde{u}$ verifies $\frac{\partial \tilde{u}}{\partial s} = J\left(\frac{\partial \tilde{u}}{\partial t} - X_{H_t}\right)$ (**)

$\tilde{u}(s,0) \in L_0$ and $\tilde{u}(s,1) \in \varphi_1^{-1}(L_1) = L_0$

Claim: $\gamma(s)$ Morse trajectory of f then $\tilde{u}(s,t) = \gamma(s)$ satisfies (**)

Claim 2 Conversely, given $u \longrightarrow \tilde{u} \longrightarrow \gamma$ gradient trajectory

Introduction to the Fukaya category

Cours 2: 28/06/16

Text: Auroux, A beginner's introduction to Fukaya categories

Lec 1 $\subset$ [Au, §1]
Lec 2 $\subset$ [Au, §2] $\cup$ {examples}
Lec 3 $\subset$ [Au, §4]

Commentary on Lecture 1:

Why $\mathbb{R} \circlearrowright \hat{\mathcal{M}}(p, q, \beta, J_t)$ is proper. $(a\cdot u)(s,t) := u(s+a, t)$

Lemma: $a \cdot u = u \iff a = 0$ or $u =$ const

Pf: $\lim_{k\to\pm\infty} u(s_0 + ka, t) = p = u(s_0, t)$ ☒

Lemma: $u_i \to u$ and $a_i \cdot u_i \to u \iff a_i \to 0$ or $u =$ const

Cor: $(\hat{\mathcal{M}} \setminus \{\text{const}\})/_{\mathbb{R}}$ is a mfd of dim $i(\beta) - 1$

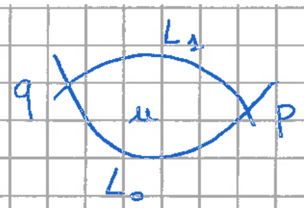
⑥ Grading

Def: $\mathcal{G}(n) :=$ {linear Lag subspaces $L \subset (\mathbb{C}^n, \omega_{std})$}

Lemma: $\mathcal{G}(n) \cong U(n)/O(n) \Rightarrow \pi_1(\mathcal{G}(n)) \underset{\mu = \text{Maslov index}}{\cong} \mathbb{Z}$

Extend to $\mu : P\mathcal{G}(n) \longrightarrow \mathbb{Z}$
$P\mathcal{G}(n) =$ {continuous maps $\varrho : [0,1] \longrightarrow \mathcal{G}(n)$ s.t. $\varrho(0) \pitchfork \varrho(1)$}

μ = unique map s.t.
- continuous map
- $\mu(\varrho_1 \times \varrho_2) = \mu(\varrho_1) + \mu(\varrho_2)$ with $\mathcal{G}(n_1) \times \mathcal{G}(n_2) \longrightarrow \mathcal{G}(n_1 + n_2)$
- $\mu(e^{i\pi N t}) = \lfloor N \rfloor + 1$ in 1-dim case



$u^*TM \cong \mathbb{R}_s \times [0,1] \times \mathbb{C}^n$

Choose e_p from T_pL_0 to T_pL_1 and e_q from T_qL_0 to T_qL_1.

Concatenate $\longrightarrow$ $\tilde{e} : S^1 \to \mathcal{G}(n)$

Fact: $i(\beta) = \mu(\tilde{e}) - \mu(e_p) + \mu(e_q)$ (does not depend on choice because if $e_1 \in P\mathcal{G}$, $e_2 \in \pi_1 \mathcal{G}$, $\mu(e_1 \# e_2) = \mu(e_1) + \mu(e_2)$)

$\dim \mathcal{M}(p, q, \beta, J_t)$

Consider
$$\begin{tikzcd} \mathcal{G}(n) \arrow[r, hook] & \mathcal{G}M \arrow[d] \\ & M \end{tikzcd}$$
$\mathcal{G}M \leftarrow$ Lag subspaces of TM

Let
$$\begin{tikzcd} \tilde{\mathcal{G}}(n) \arrow[r, hook] & \tilde{\mathcal{G}M} \arrow[d] \\ & M \end{tikzcd}$$
be a fiberwise universal cover of $\mathcal{G}M$
(exists $\iff 2c_1(TM) = 0$)
(need not be unique)

Def: A grading of $L \subset M$ is a lift
$$\begin{tikzcd} & \tilde{\mathcal{G}M} \arrow[d] \\ L \arrow[r] \arrow[ur, dashed] & \mathcal{G}M \end{tikzcd}$$

(exists if $\pi_1(L) \to \pi_1(\mathcal{G}M) \to \mathbb{Z}$ vanishes) $\leftarrow$ classifies $\tilde{\mathcal{G}M}$

$\mu_L \in H^1(L, \mathbb{Z})$ Maslov class of L

Suppose L_0, L_1 are equipped with gradings $\forall p \in L_0 \cap L_1$
$\exists!$ homotopy class of path $e_p : T_pL_0 \to T_pL_1$ that lifts to a path $\tilde{T_pL_0} \to \tilde{T_pL_1}$

Def: $|p| = \mu(e_p) \Rightarrow \mathbb{Z}$-grading on $CF(L_0, L_1) = \Lambda\langle L_0 \cap L_1 \rangle$

Lem: $|\partial(p)| = |p| + 1$

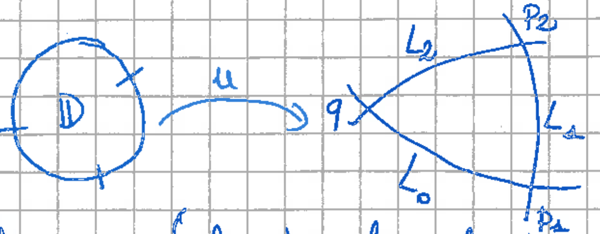
Pf: $i(\beta) = \mu(\tilde{e}) - \mu(e_p) + \mu(e_q)$ Recall that $i(\beta) = 1$ and $\mu(\tilde{e}) = 0$ (lifts to $\tilde{\mathcal{G}}(n)$) $\boxtimes$

Rk: If L_i are spin, can orient $\mathcal{M}(p, q, \beta, J_t) \rightsquigarrow$ can count with signs $\rightsquigarrow$ work over $\Lambda_{\mathbb{K}}$ with $\mathrm{char}(\mathbb{K}) \neq 2$

II. Product structures

① Product

Counting J-hol triangles



$\bar{\partial}_J u = 0 \Leftrightarrow Du \circ j = J_z \circ du$ (almost complex structure varies on $\mathbb{D}$)

j: cpx structure on $\mathbb{D}$

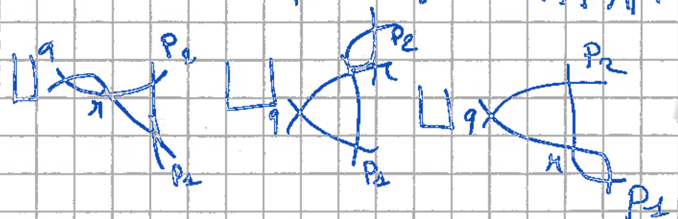
As before, have moduli space $\mathcal{M}(p_1, p_2, q, \beta, J_z)$ of such maps with $[u]=\beta$, which is a mfld of dim $i(\beta)$ for generic J_z

Def: $m_2 : CF(L_1, L_2) \otimes CF(L_0, L_1) \longrightarrow CF(L_0, L_2)$

$$m_2(p_2, p_1) := \sum_{q,\beta} \# \mathcal{M}(p_1, p_2, q, \beta, J_z)\, T^{\omega(\beta)} q$$

If L_i are graded, $i(\beta) = |q| - |p_1| - |p_2| \Rightarrow |m_2(p_1, p_2)| = |p_1| + |p_2|$

What relations do m_2 satisfy?

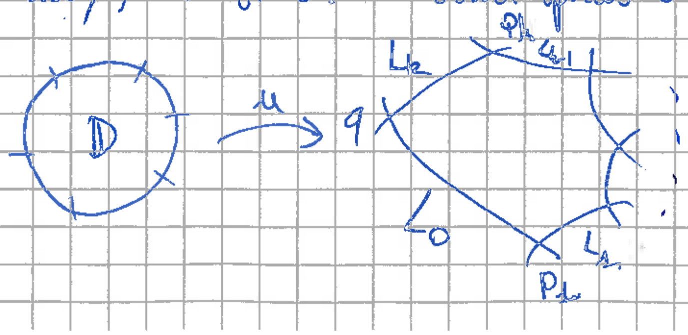
$\rightarrow$ We need to look at 1-dim component of $\mathcal{M}(p_1, p_2, q, \beta, J_z)$ which is $\mathcal{M}(p_1, p_2, q, \beta, J_z) \sqcup$ 

$\Rightarrow \partial m_2(p_2, p_1) + m_2(\partial p_2, p_1) + m_2(p_2, \partial p_1) = 0$

$\Rightarrow$ get map $HF(L_1, L_2) \otimes HF(L_0, L_1) \xrightarrow{m_2} HF(L_0, L_2)$

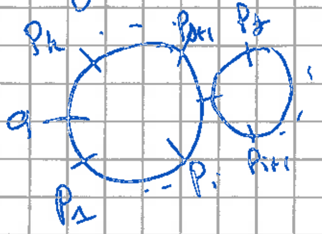
② A_∞-products

More generally, consider moduli spaces $\mathcal{M}(p_1, \dots, p_k, q, \beta, J_z)$ of hol maps



Counting 0-dim component of mod space of such J-hol maps defines a map $m_k : CF(L_{k-1}, L_k) \otimes \dots \otimes CF(L_0, L_1) \to CF(L_0, L_k)$ (of degree $2-k$ if L_i graded)

Counting boundary of 1-dim compactified moduli space gives relation



$$\Rightarrow \sum m^{k-j+i}(p_k, \dots, m^{j-i}(p_j \dots p_{i+1})\, p_i, \dots, p_1) = 0$$

$\Rightarrow$ We have defined an A_∞-category (up to defining $CF(L, L')$ when L and L' are not necessarily transverse)

Would like to define $Fuk(M, \omega)$

- Obj are $L \subset M$ Lagrangian $\omega|_{\pi_2(M,L)} = 0$ (graded, Spin)
- $Hom(L_0, L_1) = CF(L_0, L_1)$
- A_∞-structure maps m_k

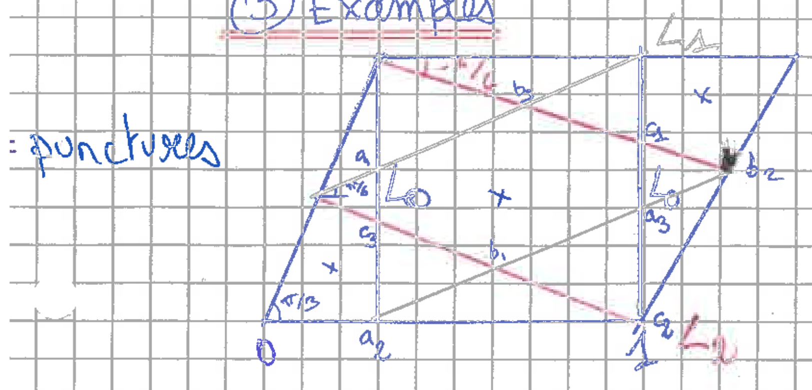
Issue: $Hom(L_0, L_1)$ only defined for $L_0 \pitchfork L_1$

$\Rightarrow$ Choose system of Ham perturbations to make $L_0 \pitchfork \psi_{01}(L_1)$. [Seidel's book]

$Hom(L_0, L_1) = \Lambda \langle L_0 \cap \psi_{01}(L_1) \rangle$ in this setup

$h(Fuk(M, \omega))$ is (deg 0 part of) Donaldson-Fukaya category

3) Examples



$\mathcal{A}$ = full subcategory of $h(Fuk(T^2 \setminus 3\,pts))$

Ex: for appropriate gradings of L_i (real $\tilde{G}M \to GM$ extends to T^2)

$HF(L_0, L_1) = \Lambda\langle a_1, a_2, a_3\rangle \quad |a_i| = 0$

$HF(L_1, L_2) = \Lambda\langle b_1, b_2, b_3\rangle \quad |b_i| = 0$

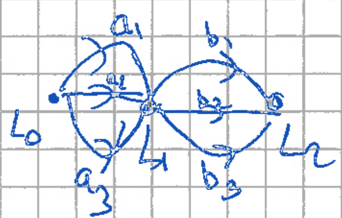
$HF(L_0, L_2) = \Lambda\langle c_1, c_2, c_3\rangle \quad |c_i| = 0$

$\Rightarrow$ morph spaces in other direction are spanned by $a_i^\vee$ in deg 1

$HF^\cdot(L_i, L_i) \cong H^\cdot(S^1) = \Lambda\langle e, \theta\rangle, \quad |e| = 0, \quad |\theta| = 1$

Ex: $m_2(b_j, a_i) = \begin{cases} c_k & \text{when } \{i,j,k\} = \{1,2,3\} \\ 0 & \text{if } i = j \end{cases}$

Rmk: $\mathcal{F}(Fuk(M,\omega))$ has identity elements $e \in H^0(L) \cong HF^0(L,L)$



$b_i a_j = b_j a_i \quad (= c_k) \text{ for } i \neq j$

$b_i a_i = 0$

Compare with $Coh(\mathbb{P}^2)$ $\quad \mathcal{O}, \Omega^1(1), \Omega^2(2)$: example of mirror symmetry.

Maydanskiy M.

Complementary course: Introduction to the Fukaya category.

Course 2: 29/06/16

Topics
(On the exercise sheet on webpage)

#1 Exact lagrangians, CF over $\mathbb{Z}_2$ ⊕ Example with $\mathbb{R}^2$ (with gradings).

#2 Gromov bubbling of spheres: some definitions + example in $\mathbb{C}P^2$

#3 Stasheff associahedra & moduli spaces of disks with marked points.

#4 Hofer displacement energy & torsion in Floer cohomology over Λ_0 (Novikov ring).

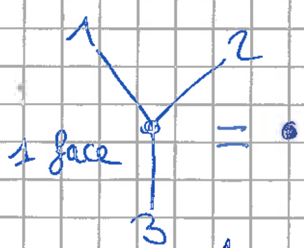
Topic #3.

Def: K_k is a polytope with

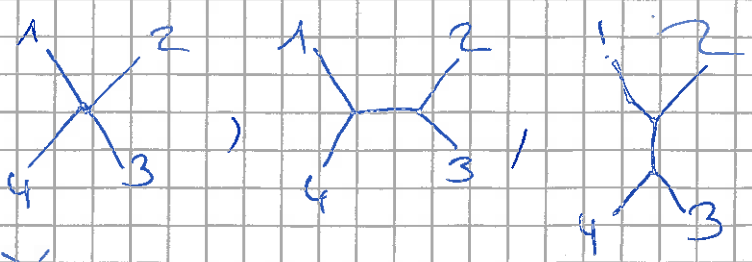
- Faces indexed by planar trees with $k+1$ labeled, cyclically ordered semi-∞ leaves (no finite leaves, all vertices deg ≥ 3) up to planar isom.
- $\text{Face}_{T_1} \subset \text{Face}_{T_2}$ if T_2 is obtained from T_1 by contracting internal edges

(alternatively if T_1 is obtained from T_2 by expanding vertices)

Example: K_2

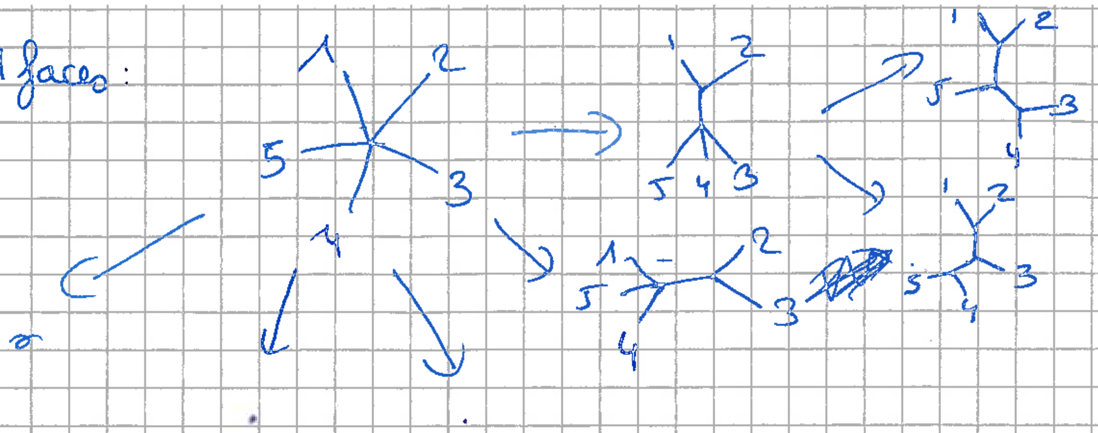


K_3: 3 faces:

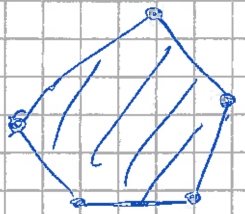


$\Rightarrow K_3 =$

$_4$ has 11 faces:



$\rightarrow$



<u>Claim</u>: K_k are isomorphic (as usual spaces) to

$\overline{\mathcal{M}}_{0,k+1}$ compactified moduli space of discs with $k+1$ bdy marked points

$$M_3 = pt = \quad \overline{M}_3 = pt$$

($\mathrm{Aut}(D^2)$ is uniquely determined by its image of 3 different points)

$$M_4 = \qquad M_4 = \text{open space}$$

$\overline{M}_4 = $ closed

<u>How to compactify M_{k+1}?</u>

Functions on M_{k+1} are cross-ratios:

$$M_{k+1} \longrightarrow \mathbb{R}_+^{\binom{k}{3}}$$

$\overset{\shortparallel}{\mathbb{R}^k/\mathrm{Stab}}$ because

$\partial D^2 = S^1 \quad \simeq$

k points ($p_{k+1} = \infty$)

Action by $\mathrm{Stab}_\infty(\mathrm{Aut}(D^2, i))$
$\shortparallel$
pos rescaling & translations

Cross ratios are functions $\mathbb{R}^k \to \mathbb{R}$ independant under the action of $\mathrm{Stab}_\infty(\mathrm{Aut}(D^2, i))$

for $0 < l < n < m \leq k$ $\qquad f_{l,n,m} = \dfrac{x_m - x_n}{x_n - x_l}$ (doesn't change under action of translation and positive rescaling)

There is $\binom{k}{3}$ different crossratios

$$i: M_{k+1} \longrightarrow \mathbb{R}_+^{\binom{k}{3}} = [0,\infty]^{\binom{k}{3}}$$

Claim 1: i is an embedding

$$x_1, \dots, x_k \sim 0, 1, \tilde{x}_3, \dots \tilde{x}_k$$

$$\Rightarrow \delta_{1,2,n} = \frac{x_n - 1}{1-0} = x_n - 1 \quad \square$$

Claim 2: $\overline{i(M_{k+1})} \cong K_k$

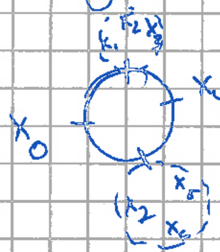
$$\vec{x}^{\,i} \in \mathbb{R}^k \qquad i(\vec{x}^{\,i}) \to p.$$

$$\vec{x}^{\,i} \to \vec{\tilde{x}}^{\,i} \text{ with } \tilde{x}^i_1 = 0 \quad \tilde{x}^i_2 = 1 \qquad \text{(fixing the gauge)}$$

$$\lim_{i \to \infty} (\tilde{x}_k)^i \to x_k.$$

It might be some x_k are equal $\to$ tells that these merge into a bubble tree. Example: $x_5 = x_6 = x_7$

$x_1 = x_2 = x_3$.



To see what happens to the group x_5, x_6, x_7 we can change the gauge such that $\tilde{x}_5 = 0$ $\quad \tilde{x}_6 = 7$ and proceed by induction

is a bubble tree: for example

Topic #4

Recall that $CF(L_0, L_1) = \bigoplus_{p \in L_0 \cap L_1} \Lambda\langle p \rangle$. with

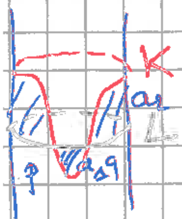
$$\partial p = \sum_{q,\beta} \# \mathcal{M}(p,q,\beta)\, T^{\omega(\beta)} q \quad \text{where } \Lambda = \left\{ \sum_{i \in \mathbb{N}} a_i T^{\lambda_i} \,/\, a_i \in \mathbb{K},\ \lambda_i \xrightarrow[i\to\infty]{} \infty \right\}$$

Novikov field

$$CF_{\Lambda_0} = \bigoplus \Lambda_0 \langle p \rangle \text{ with } \Lambda_0 = \left\{ \sum a_i T^{\lambda_i} \,/\, \begin{array}{l} a_i \in \mathbb{K} \\ \lambda_i \xrightarrow[i\to+\infty]{} \infty \\ \lambda_i \geq 0 \end{array} \right\} \text{ Novikov ring}$$

⚠ CF_{Λ_0} not invariant under Ham isotopy

Example: $M = \mathbb{C} = S^1 \times \mathbb{R} = T^*S^1$ $\qquad L = S^1 \times 0$



$$CF_{\Lambda_0}(K,L) = \Lambda_0^p \oplus \Lambda_0^q$$

$$\partial p = T^{a_1} q + T^{a_2} q = T^{a_1}\left(1 + T^{a_2 - a_1}\right) q \qquad a_2 \geq a_1$$

* If $a_1 = a_2$, $\partial p = 0$ $\quad HF_{\Lambda_0} = \Lambda_0\langle p\rangle \oplus \Lambda_0\langle q\rangle$

* If $a_2 > a_1$, $\mathrm{Ker} = \Lambda_0\langle q\rangle$, $\mathrm{Im} \simeq T^{a_1}\Lambda_0^q$ $\Rightarrow HF = \Lambda_0 / T^{a_1}\Lambda_0$

Take (M,ω) symplectic $\quad \phi \in \mathrm{Ham}^c(M,\omega)$ (compact support)

Def: • Hofer norm, $e(\phi) = \inf_H \int_0^1 (\max H_t - \min H_t)\,dt$

where H vary over all $H : M \times [0,1] \to \mathbb{R}$ s.t. $\phi_1^{H_t} = \phi$.

• $d(\varphi,\psi) = e(\varphi\psi^{-1})$ bi-inv metric on Ham^c (McDuff Lalonde)

Def (Displacement energy): $K, L \subset M$

$$e(K,L) = \inf_{\{\psi / \psi(K) \cap L = \emptyset\}} e(\psi)$$

Obs $\quad e(K,L) = e(L,K)$

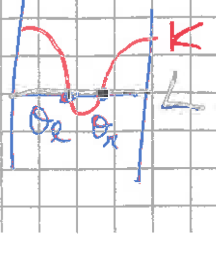
Question: What is $e(L,K)$ in the example?

Need to show: ① Find a Ham φ s.t. $e(\varphi) = a_1$ and $L \cap K = \emptyset$

② Show that if φ displaces K from L, $e(\varphi) \geq a_1$

②: Suppose φ_t displaces K from L

K will end up above L



$\mathbb{C} = \mathbb{R} \times S^1$
(r, θ)

Area swept by the arc $\theta_2 - \theta_1$ is $> a_1$.

$$\phi : [0,1] \times S^1 \longrightarrow \mathbb{C}$$
$$(t, \theta) \longmapsto \varphi_t(p_\theta^t)$$

area $\quad a_1 \leqslant \displaystyle\int_{[0,1]\times[\theta_2,\theta_1]} \phi^* \omega = \int (dt \wedge d\theta)\Big(\iota X_{H_t}, \phi_* \tfrac{\partial}{\partial\theta}(\theta)\Big)$ $\quad$ ($X_{H_t} = \phi_* \frac{\partial}{\partial t}$)

$$= \int_{[0,1]} \left(\int_{\theta_2}^{\theta_1} dH_t(\,)\, d\theta \right) dt \leqslant \int_{[0,1]} (\max H_t - \min H_t)\, dt = e(H_\cdot$$

wrong argument. It seems there are no easy way to prove this bound

Thm $\boxed{5}$ in F000 (Anomaly and obstruction): proved this result in full generality

→ See N. Sheridan's webpage to have a PDF solution which involves an energy-capacity inequality in $\mathbb{R}^4$

Introduction to the Fukaya category.

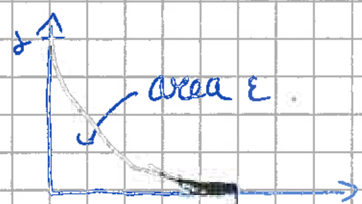
Cours 3 : 30/06/16

III. Triangulated structure

Recall (M, ω) - symplectic manifold
$\rightsquigarrow \mathrm{Fuk}(M,\omega)$ A_∞-category.
Objects = Lagrangians $L \subset M$ s.t. $\omega|_{T(M,L)} = 0$.
From this A_∞-category, we can construct $\mathrm{Tria}(\mathrm{Fuk}(M,\omega))$

① Lagrangian connect sum & mapping cones

Let $f_\varepsilon :]0,\delta] \to [0,\delta]$ have graph



$L_\varepsilon := \left\{ p_i = \frac{\partial}{\partial q_i}\left(f_\varepsilon(\|q\|)\right) \right\} \subset (\mathbb{R}^n \times \mathbb{R}^n, \omega_{std})$ is a Lagrangian

Let $L_1, L_2 \subset M$ be Lagrangian, $p \in L_1 \cap L_2$ transverse intersection point. Identify nbhd of p with $B_\delta(0)$ s.t.
$L_1 \mapsto \mathbb{R}^n \times 0$ and $L_2 \mapsto 0 \times \mathbb{R}^n$

$$L_\varepsilon \setminus B_\delta(0) \cong (\mathbb{R}^n \times 0 \cup 0 \times \mathbb{R}^n) \setminus B_\delta(0)$$

Glue in a copy of $L_\varepsilon \rightsquigarrow$ get $L_1 \#_\varepsilon L_2$
If $L_1 \cap L_2 = \{p\}$, $L_1 \#_\varepsilon L_2$ is embedded
$\Rightarrow$ new object of $\mathrm{Fuk}(M,\omega)$

Claim: In $\mathrm{Tria}(\mathrm{Fuk}(M,\omega))$, $L_1 \#_\varepsilon L_2 \cong \mathrm{Cone}(L_2 \xrightarrow{T^{-\varepsilon} p} L_1)$
It follows from (unpublished) work of FOOO.

Lem: Let L_3 be another Lagrangian. Assume $L_1 \cap L_2 \cap L_3 = \emptyset$
Then for $\varepsilon > 0$ small $L_3 \cap (L_1 \#_\varepsilon L_2) = (L_3 \cap L_1) \cup (L_3 \cap L_2)$

$\Rightarrow \mathrm{Hom}(L_3, L_1 \#_\varepsilon L_2) \cong \mathrm{Hom}(L_3, L_1) \oplus \mathrm{Hom}(L_3, L_2)$

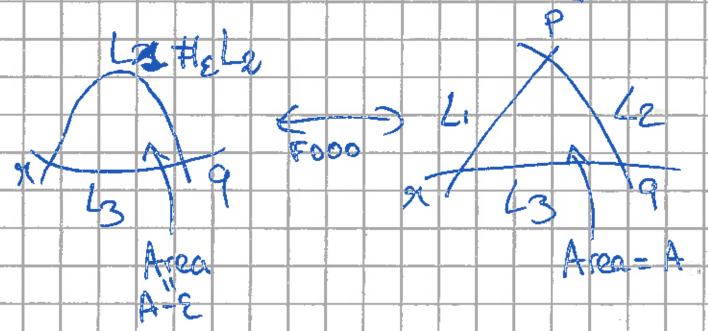
Claim. $\left(\mathrm{Hom}(L_3, L_1 \#_\varepsilon L_2), m_1\right) \cong \left(\mathrm{Hom}_{Tw(Fuk)}(L_3, L_2 \xrightarrow{T^{-\varepsilon}p} L_1), m_1^{Tw(Fuk)}\right)$

Let $q, r \in L_3 \cap (L_1 \#_\varepsilon L_2)$. FOOO's result says (roughly): the space of J-holomorphic discs w/ bdy on $L_1 \#_\varepsilon L_2$ is isom. to the space of J-hol discs w/ bdy on L_1 and L_2, which are allowed to "switch" from L_2 to L_1 at p (but not bac from L_1 to L_2).

For strips:

$q \in$	$r \in$	$\mathcal{M}(q,r)_{L_1 \#_\varepsilon L_2}$
$L_3 \cap L_1$	$L_3 \cap L_1$	$\mathcal{M}(q,r)_{L_1}$
$L_3 \cap L_2$	$L_3 \cap L_2$	$\mathcal{M}(q,r)_{L_2}$
$L_3 \cap L_1$	$L_3 \cap L_2$	$\emptyset$
$L_3 \cap L_2$	$L_3 \cap L_1$	$\mathcal{M}(q,p,r)_{L_1,L_2}$

where $\mathcal{M}(q,p,r)_{L_1,L_2}$



$\Rightarrow m_1$ is given by:

$$\begin{tikzcd} \mathrm{Hom}(L_3, L_2) \arrow[loop below, "m_1"] \arrow[r, phantom, "\oplus"] \arrow[r, bend right=40, "m_2(T^{-\varepsilon}p, -)"'] & \mathrm{Hom}(L_3, L_1) \arrow[loop below, "m_1"] \end{tikzcd}$$

(and A_∞-products $m_k^{Tw(Fuk)}$ work similarly.

$$m_k^{Tw(Fuk)} = \sum_{t=0}^{\infty} \sum_{i_0 + \dots + i_k = t} m_k(\delta^{i_0}, -, \delta^{i_1}, -, \dots, \delta^{i_k})$$

② (Split) generation

Def: The objects $L_1 \dots L_k$ generate an A_∞ cat $\mathcal{A}$ if all objects of $\mathcal{A}$ are isomorphic in $Tw(\mathcal{A})$ to a twisted complex built from L_i.

They split-generate if every object of $\mathcal{A}$ is isomorphic in $Tw(\mathcal{A})$ to a direct summand of such

E.g. $M = T^2 = \mathbb{R}^2/\mathbb{Z}^2$ $\quad L_1 = \{x=0\}$, $\quad L_2 = \{y=0\}$

Ex You can get curves in any homology class of T^2 by taking iterated Lag connect sums of L_1 and L_2.

But they do not generate $Fuk(M)$: let $\theta \in \Omega^1(T^2 \setminus \{\tfrac{1}{2},\tfrac{1}{2}\})$

s.t. $d\theta = \omega$, $\theta|_{L_1} = 0 = \theta|_{L_2}$

For any $\left(\bigoplus L_i[n_i], \delta_{ij}\right) \in Tw(Fuk)$, set

$$G\left(\bigoplus_i L_i[n_i], \delta_{ij}\right) := \sum (-1)^{n_i} \int_{L_i} \theta \in \mathbb{R}/\mathbb{Z}$$

Prop: (Abouzaid) If $\mathcal{L}$, K are isomorphic twisted complex then $G(\mathcal{L}) = G(K)$

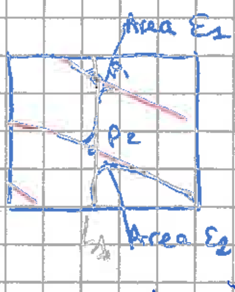
Further $G(\mathrm{Cone}(\mathcal{L} \xrightarrow{\delta} K)) = G(K) - G(\mathcal{L})$

Intuition: In Homology class , there is a S^1-family of Lagrangian non hamiltonian isomorphic

G represents this family: 

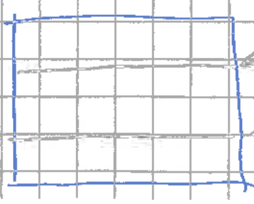
Cor: L_1 & L_2 generate the subcat of $Fuk(M)$ consisting of balanced curves (those with $\int_L \theta = 0$)

Claim: L_1 & L_2 split-generate $Fuk(M)$



$L_3 = \{y = -\tfrac{1}{2}x\}$

$\mathrm{Cone}\left(L_3 \xrightarrow[T^{-\varepsilon_1}p_1 + T^{-\varepsilon_2}p_2]{} L_1\right)$?

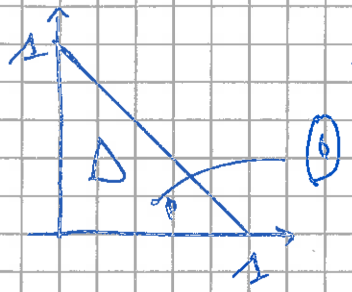


$\leftarrow \{y = \tfrac{1}{4} + \varepsilon_2 - \varepsilon_1\}$ $\longrightarrow$ split-generated by L_1 & L_2

$\leftarrow \{y = \tfrac{1}{4} + \varepsilon_1 - \varepsilon_2\}$

Using $\mathcal{E}_1, \mathcal{E}_2$, we can split-generate curves with any value of $\int_L \theta$ in this homology class ☒

③ Examples in $\mathbb{CP}^2$.



$\pi: \mathbb{CP}^2 \longrightarrow \Delta$ $\quad L_p = \pi^{-1}(p)$ Lagrangian torus fiber.

Claims: ① $Fuk(\mathbb{CP}^2) = \coprod_{\lambda \in \Lambda} Fuk(\mathbb{CP}^2)_\lambda$

(because $\omega(\pi_2(M)) \neq 0 \Rightarrow \omega|_{\pi_2(M,L)} \neq 0$)

② $L_p \in Fuk(\mathbb{CP}^2)_{\lambda(p)}$ $\quad L_p \cong 0$ unless $p = (\frac{1}{3}, \frac{1}{3})$

(where $\lambda(p_1, p_2) = T^{p_1} + T^{p_2} + T^{1-p_1-p_2}$)

③ Thm (Abouzaid-FOOO). $L_{(1/3, 1/3)}$ split-generates $Fuk(\mathbb{CP}^2)_{3T^{1/3}}$ if we work over $\Lambda_{\mathbb{C}}$ (which we can work over because T^2 is spin)

Note: $H_2 \neq 0$ for $L = L_p \rightsquigarrow Fuk$ not $\mathbb{Z}$-graded but has a $\mathbb{Z}/2$-graded

④ $HF(L_{(1/3,1/3)}) \cong Cl_2(\Lambda_{\mathbb{C}})$ ← Clifford algebra

$\qquad \cong Mat_{2,2}(\Lambda_{\mathbb{C}})$

$\Rightarrow L$ has "formal direct summand" K, $Hom(K,K) \cong \Lambda_{\mathbb{C}}$ and K also split-generates $Fuk(\mathbb{CP}^2)_{3T^{1/3}}$

$\Rightarrow D^\pi Fuk(\mathbb{CP}^2)_{3T^{1/3}} \cong D^b(\Lambda_{\mathbb{C}})$

Idempotent completion