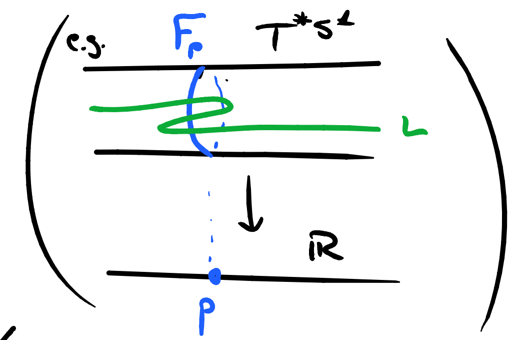
Jingyu Zhao – Complementary Lecture

Thursday, July 7, 2016 2:28 PM

(I) Convergence of the differential in family Floer theory

Recall: Given a Lagn torus fibration $\begin{array}{c} X \\ \pi\downarrow \\ Q \end{array}$ and an

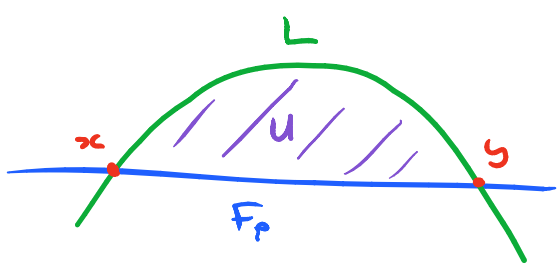


unobstructed Lagn L in X

($\exists$ a.c.s. J s.t. $\nexists$ J-holo^c disc w/ ∂ on L)

$$CF^*(F_p, L) \cong \mathbb{Z}\langle F_p \pitchfork L \rangle$$

The usual diff'l



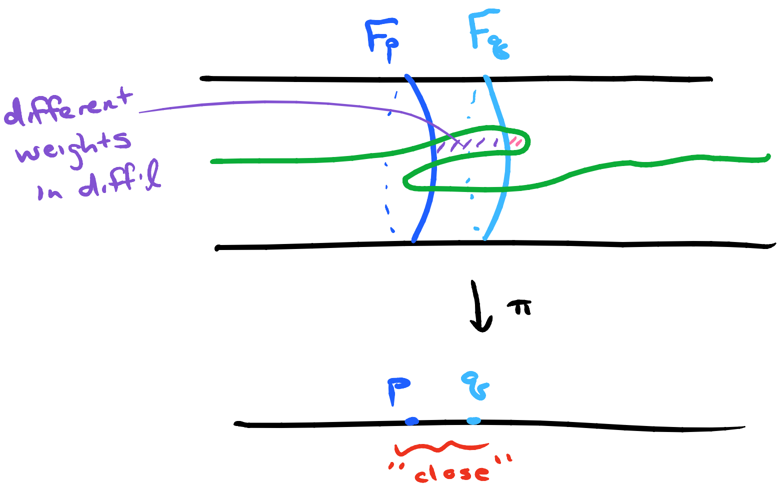
weighted by

$$T^{\omega(u)} \cdot \cancel{\mathrm{hol}_\nabla(\partial u)}$$

We'll ignore this for this talk.
Won't change arguments.

Q: How to relate $CF^*(F_p, L)$ and $CF^*(F_\xi, L)$?

Q: How to relate $CF^*(F_p, L)$ and $CF^*(F_q, L)$?



(When p is "close to" q)

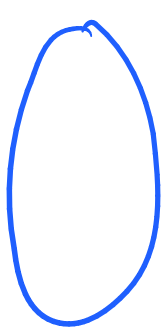
A: $CF^*(F_p, L) \cong CF^*(F_q, L)$

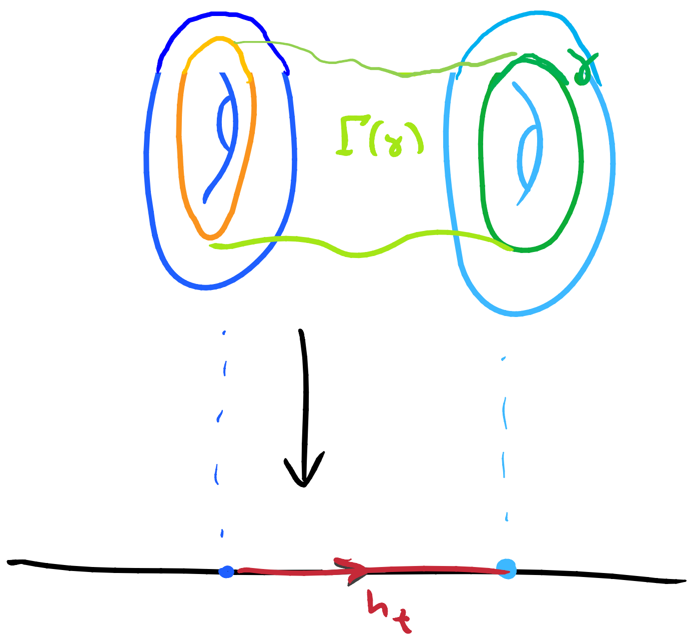
as vector spaces

But the differential changes

① Analyze the area change.

- Fix a fibre F_p, $\exists$ a map from a ("small") nbhd U_p of $p \in Q$ to a nbhd of zero in $H^1(F_q; \mathbb{R})$





$\Gamma(\gamma)$ via parallel transport.

The map is

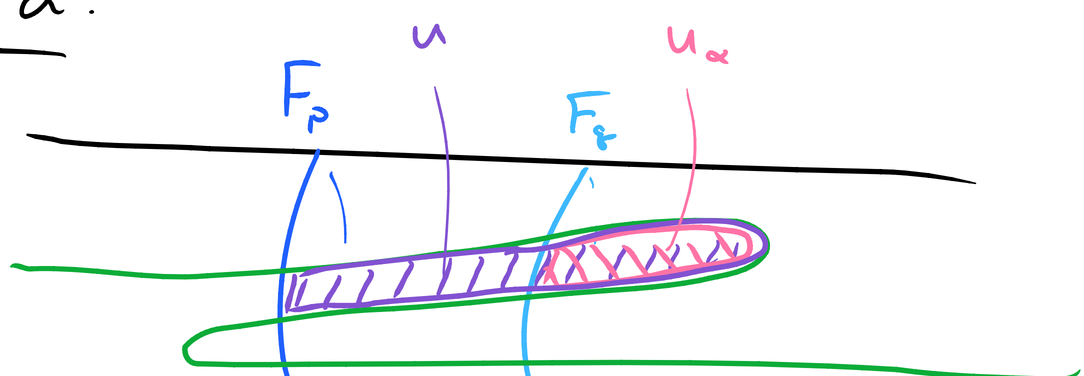
$$q \in U_p \longmapsto \alpha : \gamma \mapsto \int_{\Gamma(\gamma)} \omega$$

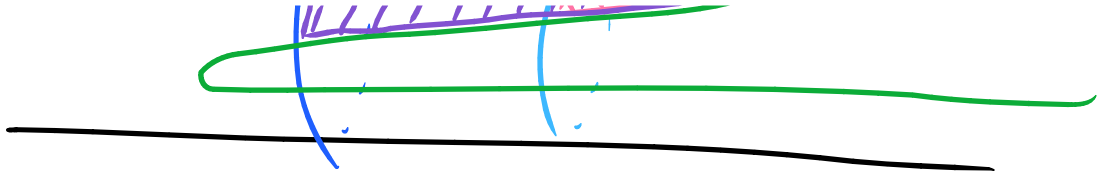
Claim: This map is a local diffeom.

$$T_pQ \cong H^1(F_q;\mathbb{R})$$

$$v = \frac{dh_t}{dt}\Big|_{t=0} \longmapsto \underbrace{i_{X_t}\omega}_{=\dot{\alpha}} \quad \left(X_t = \frac{dj_t}{dt},\ j_t : T^n \to X\right)$$

Claim 2:





$$\int u_\alpha^* \omega - \int u^* \omega = \langle \alpha, [\partial u] \rangle$$

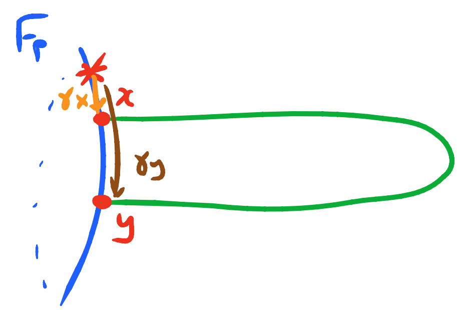
+ minor error

i.e. $$\left| \int u_\alpha^* \omega - \int u^* \omega - \langle \alpha, [\partial u] \rangle \right| < C_\alpha$$

independent of u

Here, $[\partial u] = \gamma_y^{-1} * \partial u|_{F_p} * \gamma_x$

(is nullhomologous)



Diff'l in Family Floer homology:

$$\partial_{x,y} := \sum_{u \in M^0(x,y)} T^{\int u^* \omega} \cdot z^{[\partial u]}$$

Choose basis $H^1(F_p;\mathbb{Z}) \cong \mathbb{Z}^n$

dual basis $H_1(F_p;\mathbb{Z}) \cong \mathbb{Z}^n$

Then $[\partial u] = (i_1, \dots, i_n) \in \mathbb{Z}^n = H_1(F_p;\mathbb{Z})$

$$\left(\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n = H^1(F_p;\mathbb{R}) \right)$$

and

$$z^{[\partial u]} = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$$

<u>Upshot</u>: $d_\alpha := \sum_{u_\alpha \in \mathcal{M}^0(x,y)} T^{\int u_\alpha^* \omega}$ can be obtained from $\partial_{x,y}$ by specializing/evaluating at $z = T^\alpha$ ($\underline{\underline{\text{i.e.}}}$ $z_i = T^{\alpha_i}$).

$\underline{\underline{\text{e.g.}}}$ $\alpha = 0$, $z = 1 \implies \partial_{x,y} = d$

$^1($

$$\begin{tikzcd} X \arrow[dr, "\pi"'] & & Y = \coprod_{p \in Q} H^1(F_q, \mathcal{U}_\Lambda) \arrow[dl] \\ & Q & \end{tikzcd}$$

$$\begin{array}{c} H^1(F_p, \Lambda^*) \cong \Lambda^*[z_i^{\pm 1}] \\ \big\downarrow \mathrm{val} \\ H^1(F_p, \mathbb{R}) \end{array}$$

$\sum_{i=0}^{\infty} a_i T^{\lambda_i}$, $a_0 \neq 0$, $\lambda_0 < \lambda_1 \leq \cdots$

$\mapsto \lambda_0$

$$\mathcal{U}_\Lambda := \mathrm{val}^{-1}(0)$$

Then $z^{[\partial \mathcal{U}]}$ is a function on $Y_P = \coprod_{p \in P} H^1(F_p, \mathcal{U}_\Lambda)$

where $P \subseteq Q$ integral affine polygon

Claim: $\partial_{x,y}$ is analytic on Y_P

Problem: $\partial_{x,y}$ may not converge!

(If it does, convergence radius?)

Note:

$0 \to \mathcal{U}_\Lambda \longrightarrow \Lambda^* \xrightarrow{\mathrm{val}} \mathbb{R} \to 0$ splits

$Y_P \subseteq H^1(F_p, \Lambda^*)$

If $\alpha = 0$, $z = 1$, $\partial_{x,y} = d$ on $CF^*(F_p, L)$

due to Gromov compactness.

$\Lambda^* \xrightarrow{\text{val}} \mathbb{R}$ forms a norm $|\cdot| := e^{-\text{val}(\cdot)}$ on Λ^*

Facts :

1. val is non-archimedean norm

   $\overset{\text{i.e.}}{=}$ $|x+y| \leq \max(|x|, |y|)$

2. $\sum_i a_i T^{\lambda_i}$ converges

$$\Longleftrightarrow$$

$$|a_i T^{\lambda_i}| \to 0$$

$$\Longleftrightarrow$$

$$\text{val}(a_i T^{\lambda_i}) = \lambda_i \longrightarrow +\infty$$

as $i \longrightarrow +\infty$

Gromov: For E, $\exists$ finitely many $\lambda_i < E$ s.t. $a_i \neq 0$

$\Longrightarrow$ $\alpha = 0$, $z = 1$, $\partial_{x,y}$ converges

$$\coprod_{p \in P} H^1(F_p, \mathcal{U}_\Lambda) \qquad H^1(F_p, \Lambda^*)$$

$$\downarrow \text{val}$$

$$\alpha \in H^1(F_p; \mathbb{R})$$

Solution to problem (Fukaya's trick)

Propn: $\exists$ nbhd $\mathcal{U}_p \ni p$ s.t. $\forall\, p' \in \mathcal{U}_p$,

we pick a diffeo^m $\psi_{p'} : X \circlearrowleft$ sending F_p to $F_{p'}$ s.t.

1. $\psi_{p'}(L) = L$
2. $\psi_{p'}$ preserves tameness of a.c.s. J_p on $CF^*(F_p, L)$

(Key: Tameness is an open condition)

Pr p $\Longrightarrow$ Using $(\psi_{p'})_* J_p$ tame

Propn $\longrightarrow$ using $(\Upsilon_{p'})_* J_p$,

$$CF^*(F_p, L) \cong CF^*(F_{p'}, L)$$