Contact non-squeezing via generating functions

A low-tech proof in the language of persistence modules

Maia Fraser

University of Ottawa

fraser.maia@gmail.com

Abstract

Persistence modules are simply functors from a poset (often $\mathbb{R}$) to Vect. They have been used in symplectic geometry going back to the symplectic capacity of Viterbo and Floer-theoretic analogs (Oh, Schwartz) though the terminology of persistence, barcodes etc... arose later in Computer Science when persistence modules were studied as tools for topological data analysis. In particular Viterbo's symplectic capacity of domains in $\mathbb{R}^{2n}$ and Sandon's contact capacity of domains in $\mathbb{R}^{2n} \times S^1$ are persistences of certain homology classes in the persistence module formed by generating function (GF) homology groups. While Sandon's capacity $c_S(-)$ allows to re-prove non-squeezing of $B(R) \times S^1$ into itself for integral R (a result due to Eliashberg-Kim-Polterovich 2006), by introducing new filtration-decreasing morphisms between GF homology groups one can set up a functor from a sub-category of $\mathcal{D} \times \mathbb{Z}$ to Vect, where $\mathcal{D}$ is the category of bounded domains with inclusion. Persistences in this persistence module then yield a sequence $m_\ell(-)$, $\ell \in \mathbb{N}$ of integer-valued contact invariants for prequantized balls, such that $m_1 = c_S - 1$ and $m_\ell(B(R) \times S^1)$ is the greatest integer strictly less than ℓR. This provides an alternate proof of non-squeezing at large scale, i.e. of $B(R) \times S^1$ into itself for any $R > 1$ (proved by Chiu 2014).

Squeezing and non-squeezing

Setting: $\mathbb{R}^{2n} \times S^1$ with contact structure $\ker(dt - \alpha_L)$, Liouville form $\alpha_L = \frac{1}{2}(ydx - xdy)$ on $\mathbb{R}^{2n}$, $x, y \in \mathbb{R}^{2n}$, $t \in S^1$. Let $\mathrm{Cont}_0(\mathbb{R}^{2n} \times S^1)$ denote the identity component of the group $\mathrm{Cont}(\mathbb{R}^{2n} \times S^1)$ of compactly supported contactomorphisms.

Definition 1. A *squeezing*, resp. *coarse squeezing* of an open set U_1 into an open set U_2 is $\psi \in \mathrm{Cont}_0(\mathbb{R}^{2n} \times S^1)$, resp. $\psi \in \mathrm{Cont}(\mathbb{R}^{2n} \times S^1)$ such that $\psi(\mathrm{Closure}(U_1)) \subset U_2$.

Theorem 2. *[Eliashberg-Kim-Polterovich 06] Let $R < 1$. Then there is a contact squeezing of $\widehat{B}(R)$ into itself. By contrast, if $R \geq 1$ is an integer, there is no coarse contact squeezing of $\widehat{B}(R)$ into itself.*

Theorem 3. *[Chiu 14] Let $R \geq 1$. Then there is no contact squeezing of $\widehat{B}(R)$ into itself.*

Theorem 3'. *[F. 15] Let $R \geq 1$. Then there is no coarse contact squeezing of $\widehat{B}(R)$ into itself.*

Aim: sketch an alternate generating function-based proof of Theorem 3. Note: GF homology groups are only defined in Euclidean space and close cousins such as $\mathbb{R}^{2n} \times S^1$ so the argument we propose here is specific to this simple setting, unlike the methods of [Chiu 14] or [F. 15] (microlocal analysis, resp. contact homology) which have more potential to generalize.

Generating function homology groups

To review both Traynor and Sandon's GF homology theories we describe both before restricting to the contact setting. Let $\mathcal{I} = \mathbb{Z} \setminus \{0\}$ and $\mathcal{G} = \mathrm{Cont}_0(\mathbb{R}^{2n} \times S^1)$ in contact setting. Let $\mathcal{I} = \mathbb{R} \setminus \{0\}$ and $\mathcal{G} = \mathrm{Ham}(\mathbb{R}^{2n})$ in the symplectic setting.

Proposition 4. *[Traynor 94, Sandon 09] Let $\mathcal{U} \subset \mathcal{V} \in \mathcal{D}$. Then, for every $a \in \mathbb{N}$ there is a well-defined morphism of $\mathbb{Z}$-graded vector spaces $\iota^* : G_*^{(a,\infty]}(\mathcal{V}) \to G_*^{(a,\infty]}(\mathcal{U})$. In addition, for each open domain $\mathcal{W}$ and $\psi \in \mathcal{G}$ there is an induced isomorphism $\psi^* : G_*^{(a,\infty]}(\psi(\mathcal{W})) \to G_*^{(a,\infty]}(\mathcal{W})$ which behaves well under composition and such that the morphisms ψ^* and ι^* commute, making $G_*^{(a,\infty]}(\cdot)$ a $\mathcal{G}$-invariant functor from $\mathcal{D}$ to $\mathbb{Z}$-graded vector spaces.*

Construction in brief: The groups $G^{(a,\infty]}(\phi)$ are defined as relative homology groups,

$$G_*^{(a,\infty]}(\phi) := H_*(E, \{S \leq a\}),$$

where $S : E = \mathbb{R}^m \times \mathbb{R}^K \to \mathbb{R}$, $K \in \mathbb{N}$ is a generating function for an exact Lagrangian in $T^*\mathbb{R}^m$, resp. Legendrian in $J^1\mathbb{R}^m$, *associated to* ϕ (this association involves identifying $T^*M \times T^*M$ with $T^*(M \times M)$ but can be done for $M = \mathbb{R}^{2n}$ resp. symplectization of $\mathbb{R}^{2n} \times S^1$). Key properties:

1. there exist generating functions [Laudenbach-Sikorav 85, Sikorav 87, Chekanov 95]
2. there are many generating functions for a given ϕ but the relative homology groups $H_*(E, \{S \leq a\})$ depend only on ϕ [Theret 99], and
3. given $\phi_1 \preceq \phi_2$ there are respective generating functions S_1, S_2 for ϕ_1, ϕ_2 defined on a common E such that $S_1 \leq S_2$.

Existing morphisms and invariants of domains: Given $\phi_1 \preceq \phi_2$ the corresponding inclusion of pairs $(E, \{S_2 \leq a\}) \subset (E, \{S_1 \leq a\})$ induces morphisms on relative homology, hence *in principle* (not proved in [Traynor 94, Sandon 09] but can be handled by using paths instead of time-1 maps), well-defined **monotonicity morphisms** $\iota^* : G_*^{(a,\infty]}(\phi_2) \to G_*^{(a,\infty]}(\phi_1)$ $\rightsquigarrow$ the groups $G_*^{(a,\infty]}(\cdot)$ form a directed system over the directed set $\mathcal{H}^a(\mathcal{U})$ of (time-1 maps of) Hamiltonians supported on $\mathcal{U}$ (without a in spectrum). Define $G_*^{(a,\infty]}(\mathcal{U}) := \varprojlim_{\phi \in \mathcal{H}^a(\mathcal{U})} G_*^{(a,\infty]}(\phi)$. When $\mathcal{U} \subset \mathcal{V}$, have $\mathcal{H}^a(\mathcal{U}) \subset \mathcal{H}^a(\mathcal{V})$, thus a morphism $G_*^{(a,\infty]}(\mathcal{V}) \to G_*^{(a,\infty]}(\mathcal{U})$, also denoted ι^*. Further structure arises from inclusion of pairs of sub-level sets – this time due to filtration by level $a \leq b$ for a given generating function S rather than comparison at constant level of two generating functions $S_1 \leq S_2$. This induces **filtration-increasing morphisms** $v_a^b : G_*^{(a,\infty]}(\mathcal{U}) \to G_*^{(b,\infty]}(\mathcal{U})$. Given an isotopy $\{\psi_t\}_{t \in [0,1]} \subset \mathcal{G}$ one can show that $G^{(a,\infty]}(\psi_t \mathcal{U})$ is preserved (this requires integer a in contact setting). This **isomorphism** ψ_* induced by $\psi \in \mathcal{G}$ moreover commutes with ι^* and also v_a^b. Focus on contact setting from now on. Results for prequantized balls [Sandon 09]:

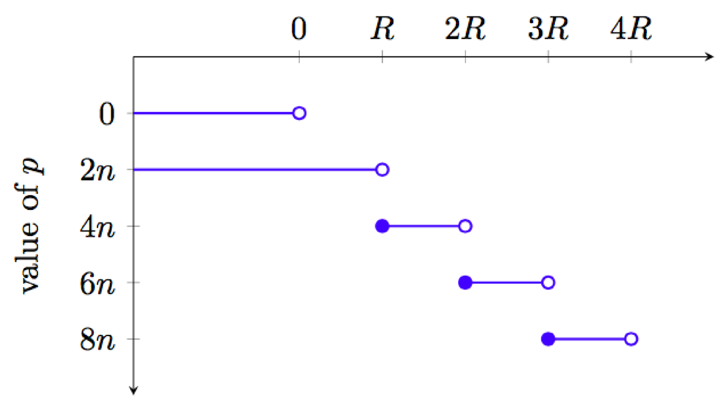

Figure 1: The solid blue lines represent for fixed p the range of a-values for which the group $G_p^{(a,\infty]}(\widehat{B}(R))$ is non-trivial (fine print: for technical reasons no groups are defined for $a = 0$ but canonical maps $G_p^{(-\infty,\infty]}(\widehat{B}(R)) \to G_p^{(b,\infty]}(\widehat{B}(R))$, $0 < b$ are non-trivial). Sandon's capacity is $\lceil d \rceil$ where d is death of "bar" (i.e. interval module) in degree $p = 2n$. Note: bars move to the right when one considers larger $\widehat{B}(R)$. Visualize this with R as third dimension: morphisms ι_* and v_{a-}^{a+} are parallel to the R- and a-axes respectively. *Unfortunately:* ι_* is contravariant, v_{a-}^{a+} covariant.

Pivots and new morphisms

Definition 5. Let $\mathcal{U} \in \mathcal{D}$ and $a \leq b \in \mathcal{I}$. When $v_a^b : G^{(a,\infty]}(\mathcal{U}) \to G^{(b,\infty]}(\mathcal{U})$ is an isomorphism, say $\mathcal{U}$ is an **[a, b]-pivot.**

By commutativity of v_a^b and ψ^*, this is a $\mathcal{G}$-invariant property. Define $\mathrm{Piv}(\mathcal{D} \times \mathcal{I})$ to be the subcategory of $\mathcal{D} \times \mathcal{I}$ (with product partial order $\subset, \leq$) having same objects but where morphisms are restricted to be compositions of $(\mathcal{U}, a) \to (\mathcal{V}, b)$ such that either $a = b$ or (when $a < b$ is strict) $\mathcal{U}$ is an $[a, b]$-pivot. By $\mathcal{G}$-invariance of pivots, $\mathrm{Piv}(\mathcal{D} \times \mathcal{I})$ is a $\mathcal{G}$-category for the $\mathcal{G}$-action in the first factor.

Proposition 6. *[F., in progress] Let $(\mathcal{U}, a) \to (\mathcal{V}, b)$ in $\mathrm{Piv}(\mathcal{D} \times \mathcal{I})$. Then there is a well-defined morphism of $\mathbb{Z}$-graded vector spaces $\delta_b^a : G_*^{(b,\infty]}(\mathcal{V}) \to G_*^{(a,\infty]}(\mathcal{U})$. When $a = b$ this coincides with the usual monotonicity morphism, i.e., $\delta_b^a = \iota^*$; when $\mathcal{U} = \mathcal{V}$ we have $\delta_b^a = (v_a^b)^{-1}$; in general the following diagram commutes:*

$$\begin{array}{ccc} G_*^{(b,\infty]}(\mathcal{U}) & \xleftarrow{\iota^*} & G_*^{(b,\infty]}(\mathcal{V}) \\ \uparrow v_a^b & \swarrow \delta_b^a & \\ G_*^{(a,\infty]}(\mathcal{U}). & & \end{array}$$

This extends and unifies the existing morphisms ι^* and $(v_a^b)^{-1}$ in such a way that $G_*^{(\cdot,\infty]}(\cdot)$ becomes a contravariant *functor from $\mathrm{Piv}(\mathcal{D} \times \mathcal{I})$ to $\mathbb{Z}$-graded vector spaces which is $\mathcal{G}$-invariant.*

A zig-zag to infinity

Lemma 7. *Let $\ell \in \mathbb{N}$ and consider lines $L_+ : y = (\ell+1)x$ and $L_- : y = \ell x$. Suppose $1 < R < R' \in \mathbb{R}$ and $a < b \in \mathbb{N}$ such that $\ell R \leq a < (\ell+1)R$ and $\ell R' \leq b < (\ell+1)R'$. Then there is a 'zig-zag' curve lying strictly below the line L_+ and on or above the line L_-, which connects the point (R, a) to (R', b) and consists of alternatingly horizontal and vertical segments at increasing y- and x-levels, such that horizontal segments occur at* integer *y-levels.*

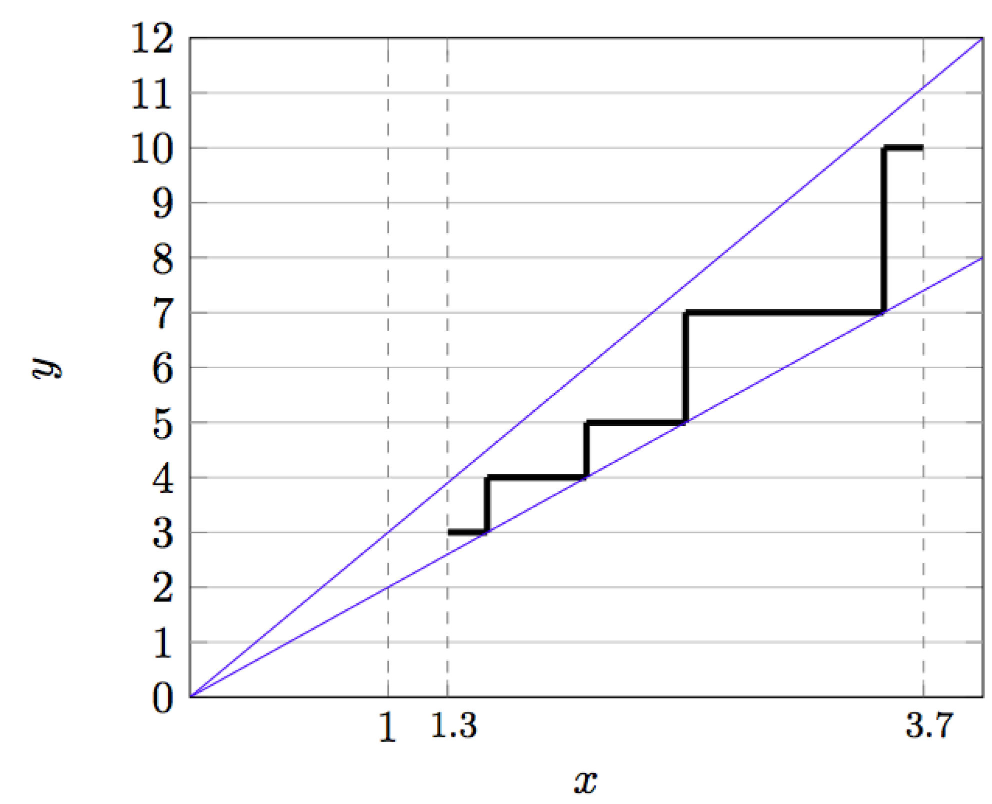

Figure 2: The lines $L_+ : y = (\ell+1)x$ and $L_- : y = \ell x$ are shown in blue for $\ell = 2$. A zig-zag as in Lemma 7 is shown in thick black for $(R, a) = (1.3, 3)$ and $(R', b) = (3.7, 10)$. Existence of a zig-zag between (R, a) and (R', b) is equivalent to existence of a morphism $(\widehat{B}(R), a) \to (\widehat{B}(R'), b)$ in $\mathrm{Piv}(\mathcal{D} \times \mathcal{I})$.

Let $\mathcal{B}$ be the subcategory of $\mathcal{D}$ whose objects are $\psi(\widehat{B}(R)) \subset \mathbb{R}^{2n} \times S^1$, $\psi \in \mathcal{G}$, $R \in (0, \infty)$. For such domains, the value R is uniquely determined though ψ is not. Write $\mathcal{B}_1 \subset \mathcal{B}$ when $R > 1$.

Lemma 8. *The domain $\widehat{B}(R)$ is an $[a, b]$-pivot if and only if $(a, b]$ does not contain ℓR for any $\ell \in \mathbb{N}$. In particular, $\mathcal{U} \in \mathcal{B}$ an $[a, b]$-pivot, $a < b \Rightarrow \mathcal{U} \in \mathcal{B}_1$.*

Sequence of invariants

Definition 9. Given $\mathcal{U} \in \mathcal{B}$ define its **signature** to be the ($\mathcal{G}$-invariant) sequence of integers $(m_\ell(\mathcal{U}))_{\ell=0}^{\infty}$ where $m_0(\mathcal{U}) := 0$ and for each $\ell \in \mathbb{N}$,

$$m_\ell(\mathcal{U}) := \sup\{a \in \mathbb{Z} : G_{2n\ell}^{(a,\infty]}(\mathcal{U}) \neq 0\}.$$

Theorem 10. *(Monotonicity) [F.] Let $\mathcal{U}, \mathcal{V} \in \mathcal{B}_1$. Then $\mathcal{U} \subset \mathcal{V} \Rightarrow m_\ell(\mathcal{U}) \subset m_\ell(\mathcal{V})$. Equivalently, non-squeezing holds for objects of $\mathcal{B}_1$.*

Proof. Let $\psi(\mathrm{Cl}\widehat{B}(R)) \subset \widehat{B}(R)$. Note $\psi \in \mathcal{G}$ fixes large $\widehat{B}(R_\dagger)$. For suitable a, b, $\delta_b^a : G_*^{(b,\infty]}(\widehat{B}(R_\dagger)) \to G_*^{(a,\infty]}(\widehat{B}(R))$ is non-trivial isomorphism, hence $\delta_b^a : G_*^{(b,\infty]}(\widehat{B}(R_\dagger)) \to G_*^{(a,\infty]}(\psi\widehat{B}(R))$ too. This factors through $\iota_* : G_*^{(a,\infty]}(\psi\widehat{B}(R)) \to G_*^{(a,\infty]}(\widehat{B}(R))$, so ι_* also an iso. Choose bad $R' = R - \epsilon$. □

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