Tilting Modules in Categories $\mathcal{O}$

Does Symplectic Duality = 3d Mirror Symmetry ?

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Category $\mathcal{O}$

By now it is well known that the good properties of the Bernstein-Gelfand-Gelfand category $\mathcal{O}$ for a semisimple Lie algebra $\mathfrak{g}$ are due its relationship with the Springer resolution. In particular, Braden, Proudfoot, Licata, and Webster have shown one can associate a similar category to any symplectic resolution $\mathcal{M}$ of an affine symplectic cone M_0 which admits a Hamiltonian $\mathbb{C}^*$-action with finitely many fixed points. More precisely, let

$$\mathcal{M}^{>} = \{m \in M \mid \lim_{t\to\infty} t\cdot m \text{ exists}\}$$

be the ascending manifold for the Hamiltonian $\mathbb{C}^*$-action and let $\widehat{\mathbb{C}[M]}$ be a deformation quantization of M. Category $\mathcal{O}$ is defined to be the category of $\widehat{\mathbb{C}[M]}$-modules supported on $\mathcal{M}^{>}$.

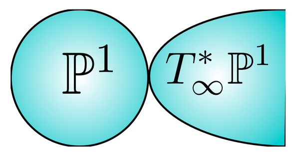

Figure 1: The ascending manifold for a Hamiltonian $\mathbb{C}^*$-action on $T^*\mathbb{P}^1$

Category $\mathcal{O}$ is always a highest weight category where the weights are fixed points of the $\mathbb{C}^*$-action which are ordered by the value attained by the real moment map. This means that for each fixed point v there is a collection of modules depicted below.

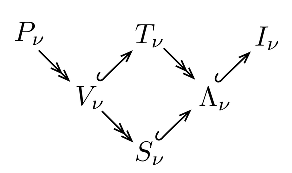

Figure 2: Modules in $\mathcal{O}$ associated to a fixed point v

The standard modules V_v and costandard modules Λ_v have composition factors consisting of simple modules $S_{v'}$ for $v' \leq v$. The projective modules P_v are admit standard filtrations, the injective modules I_v admit costandard filtrations, and the tilting modules T_v admit both.

Symplectic Duality

Empirically, each affine symplectic cone M_0 seems to have a "symplectic dual" cone $M_0^!$ satisfying the following properties. First, resolutions M of M_0 are in bijections with Hamiltonian $\mathbb{C}^*$-actions on $M_0^!$ with finitely many fixed points and vice versa. Second, the fixed points of a Hamiltonian $\mathbb{C}^*$-action on M are in bijection with the corresponding Hamiltonian $\mathbb{C}^*$-action on $M^!$. Finally the categories $\mathcal{O}$ and $\mathcal{O}^!$ are Koszul dual which behaves as depicted below.

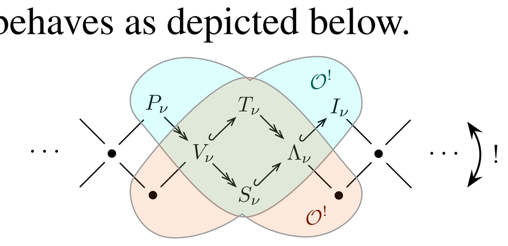

Figure 3: Koszul duality acts by reflection

Mirror Symmetry for 3d $\mathcal{N}=4$ Theories

It is the case that all known symplectic dual pairs have appeared the physics literatures as Higgs and Coulomb branches, $\mathcal{M}_H$ and $\mathcal{M}_C$, of the moduli space of vacua in 3d $\mathcal{N}=4$ supersymmetric field theories. Furthermore the 3d mirror symmetry of Intriligator and Seiberg is known to exchange the Higgs and Coulomb branches so it seems clear that symplectic duality and 3d mirror symmetry should be the same phenomenon.

To make this precise Bullimore, Dimofte, Gaiotto, and I showed that a $\mathcal{N}=(2,2)$ boundary condition $\mathcal{B}$ induces a pair of holomorphic Lagrangians $\mathcal{B}_H$ and $\mathcal{B}_C$ on the Higgs and Coulomb branches respectively. Turning on an Ω-background promotes these Lagrangians to a pair of deformation quantization modules.

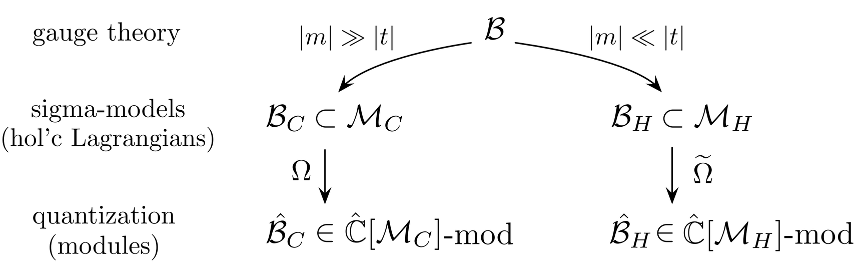

Figure 4: Boundary condtions in 3d $\mathcal{N}=4$

We also defined three families of boundary conditions, the Neumann, Dirichlet, and exceptional Dirichlet, and showed that the behave in the following way under mirror symmetry.

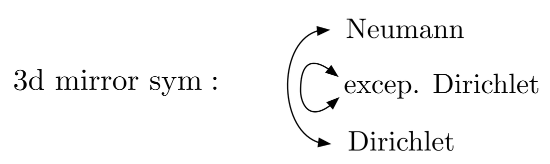

Figure 5: The action of 3d Mirror Symmetry on Boundary Conditions

In order to agree with symplectic duality we conjectured that that corresponding modules were as described in the following table.

$\mathcal{B}$	$\widehat{\mathcal{B}}_C \in \mathcal{O}_C$	$\widehat{\mathcal{B}}_H \in \mathcal{O}_H$
Neumann ($\mathcal{N}$)	tilting (T)	simple (S)
Dirichlet ($\mathcal{D}$)	simple (S)	tilting (T)
excep. Dirichlet ($\mathcal{D}_b$)	costandard (Λ)	costandard (Λ)

A Discrepancy

Unfortunately we immediately ran into a snag: the modules $\widehat{\mathcal{N}}_C$ and $\widehat{\mathcal{D}}_H$ are not tilting modules. In fact neither module is even contained in category $\mathcal{O}$. For example when $\mathcal{M}_H = T^*\mathbb{P}^1$ so category $\mathcal{O}_H$ is the BGG category $\mathcal{O}$ for $\mathfrak{sl}_2$ one of the Dirichlet boundary conditions gives rise to the nondegenerate Whittaker module. More geometrically, this corresponds to the exponential D-module e^x on $\mathbb{A}^1 \subseteq \mathbb{P}^1$ which is irregular and hence not in $\mathcal{O}_H$.

For a general theory the Lagrangians $\mathcal{N}_C$ have a particularly nice geometric description. The Coulomb branch is equipped with an integral system $\pi : \mathcal{M}_C \to \mathfrak{t}/W$. The Lagrangians of the form $\mathcal{N}_C$ are sections of this integrable systems. The indecomposable modules in $\mathcal{O}$ all quantize Lagrangians that are contained in a single fiber of π so it is clear that $\widehat{\mathcal{N}}_C$ is not in $\mathcal{O}_C$.

The Geometric Jacquet Functor

The resolution to this discrepancy is suggested by a theorem that is has been proved several times in slightly different contexts.

Theorem (Nadler, Gaitsgory-Frenkel, Campbell, Jin)**.** *Let $\mathcal{W}$ be the nondegenerate Whittaker module for $\mathfrak{g}$. Then $J(\mathcal{W})$ is the big tilting module in the BGG category $\mathcal{O}$ where J is the geometric Jacquet functor of Emerton-Nadler-Vilonen.*

The definition of the geometric Jacquet functor makes sense for any symplectic resolution $\mathcal{M}$ equipped with a Hamiltonian $\mathbb{C}^*$-action. Let $a : \mathbb{C}^* \times \mathcal{M} \to \mathcal{M}$ be the action map. Then any $\widehat{\mathbb{C}[M]}$-module M can be pulled back to a $D_{\mathbb{C}^*} \otimes_{\mathbb{C}} \widehat{\mathbb{C}[M]}$-module a^*M. Let $p : \mathbb{C}^* \times \mathcal{M} \to \mathbb{C}^*$ be the projection. Using the Kashiwara-Malgrange filtration one can make sense of the nearby cycles Ψ_p along p. Then one has $J = \Psi_p \circ a^*$.

The key observation to understand the geometric Jacquet functor is that the standards appearing in $J(\widehat{M})$ can be computed from the intersection of M with the descending manifold $\mathcal{M}^{<}$

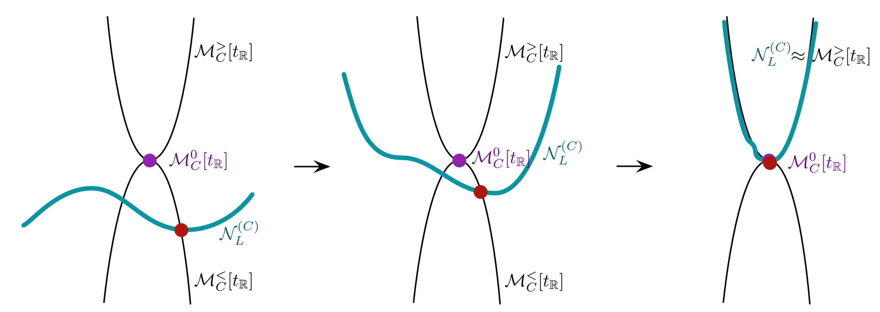

Figure 6: The behavior of the geometric Jacquet functor near an intersection with the descending manifold

The Abelian Case

For abelian gauge theories the Higgs and Coulomb branches are hypertoric varieties in the sense of Bielawski and Dancer. Just like toric varieties, their counterparts in 2d mirror symmetry, hypertoric varieties have a combinatorial nature that makes them particularly amenable to direct computation. Furthermore, abelian theories are closed under mirror symmetry so every hypertoric variety is both a Higgs and Coulomb branch. In particular this means it is enough to study either $\mathcal{D}_H$ or $\mathcal{N}_C$.

In the hypertoric case the modules $\mathcal{D}_H$ are the Hamiltonian reduction of certain well studied D-modules M_A on $\mathbb{C}^n$ known as Gelfand-Kapranov-Zelevinsky hypergeometric systems which are defined as follows. Let A be a $k \times n$ integral matrix. Then M_A is defined to be the cyclic $D_{\mathbb{C}^n}$-module with relations:

$$A \cdot \begin{bmatrix} x_1\partial_1 \\ \dots \\ x_n\partial_n \end{bmatrix} = 0 \qquad \prod_{b_i>0} \partial_i^{b_i} = \prod_{b_i<0} \partial_i^{b_i} \text{ where } b \in \ker A.$$

In my thesis I was able to use the theory of GKZ hypergeometric systems to understand the standard and costandard filtrations on $J(\mathcal{D}_H)$ and prove the following theorem.

Theorem (Hilburn)**.** *The modules $J(\mathcal{D}_H)$ and $J(\mathcal{N}_C)$ are tilting and every tilting module arises in this way. Moreover $J(\mathcal{D}_H)$ is Koszul dual to the simple module $\mathcal{D}_C$ and $J(\mathcal{N}_C)$ is Koszul dual to the simple module $\mathcal{N}_H$. Thus for abelian theories symplectic duality and 3d mirror symmetry coincide.*

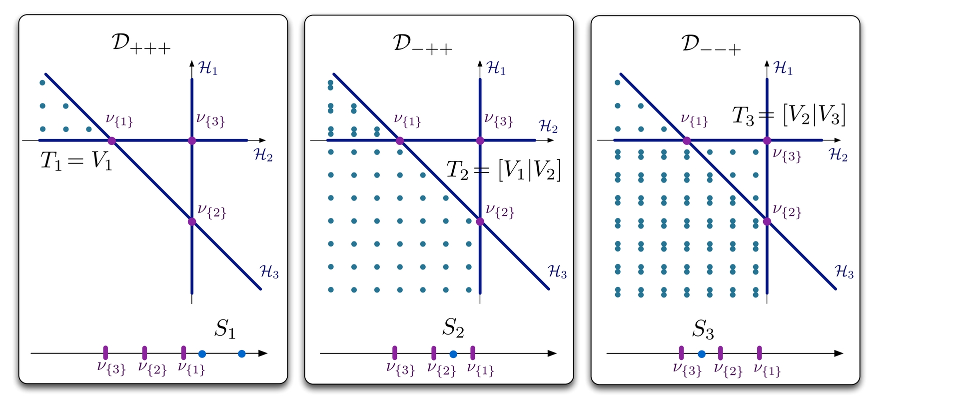

Figure 7: Dirchlet boundary conditions

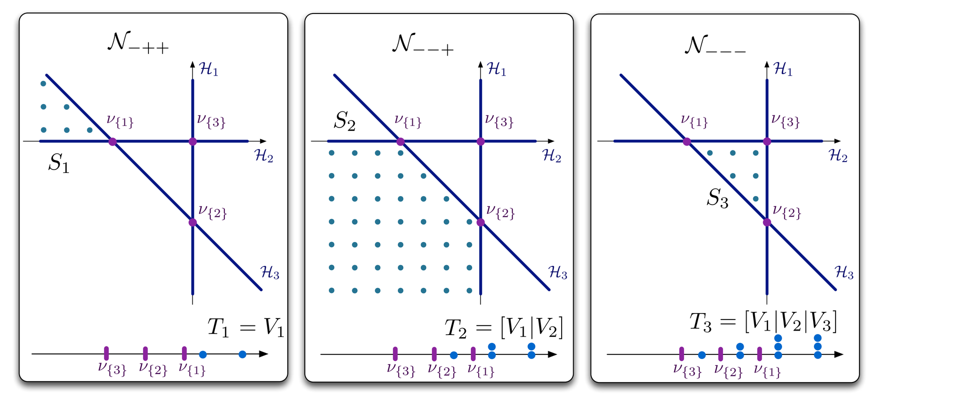

Figure 8: Neumann boundary conditions

Acknowledgements

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