Warped product Finsler manifolds and gradient fields

Parvaneh Joharinad
Institute for Advanced studies in Basic Sciences, Zanjan, Iran
p.joharinad@iasbs.ac.ir

Institute for Advanced Studies
in Basic Sciences
Gava Zang, Zanjan, Iran

Abstract

Here the Finsler metric F is used to define a hamiltonian formalism on TM_0 without F being restricted to be absolutely homogeneous or considered as a lagrangian on entire tangent bundle. This method has the potential to be generalized to special subbundles of tangent bundle, which is used to define the hamiltonian formalism related to Finsler warped metric.
Afterwards, reiterating the definition of gradient fields in Finsler geometry from [7], some basic results for Finsler manifolds obtaining a conformal gradient field are presented.

Introduction

The strong ties between conformal gradient fields and conformal transformations between Einstein spaces made the study of these special vector fields important. This goes back to Brinkman's investigation in 1920's [4], where he showed that the conformal characteristic function of such a transformation has conformal gradient. In Riemannian geometry, conformal gradient fields are essential solutions of the following differential equation

$$\nabla^2 f = (\frac{\Delta f}{n})g. \tag{1}$$

This equation has been studied by Fialkow, Yano, Obata and others. Equation (1) helps to prove that for the foliation $\mathcal{F}$ of M whose leaves are the connected components of the fibers of f, the transverse submanifolds are perpendicular to leaves and the metric tensor has warped product representation in foliated chart[8].
present poster is a part of a research project that aims to come up with a generalization of a theorem of Lichnerowicz to Finsler geometry, which states that the Riemannian manifold M^n with constant scalar curvature K is isometric to a scaled n-sphere if and only if it admits a conformal gradient field. It has been hitherto shown in [3], co-authored by B. Bidabad, that if $\nabla^2 f = \phi(x)g$ has a solution on an n-dimensional $(n \geq 2)$ simply connected compact Finsler manifold of constant Ricci curvature, then M is homeomorphic to n-sphere. The question is under which conditions this homeomorphism could enhance to an isomorphism.

Finsler structure

For considering the global aspect of Finsler geometry, we first the hamiltonian formalism on TM_0 obtained from Finsler function. The advantage of this method is that there is no need to restrict Finsler norm to be absolutely homogeneous, which eliminates a big class of interesting non Riemannian Finsler examples, or consider it as lagrangian on entire tangent bundle. Moreover, This method has the potential to be generalized to spacial sbbundles of tangent bundle.
Let $\pi : TM_0 \longrightarrow M$ be slit tangent bundle , where (M, F) is a C^∞ connected differentiable Finsler manifold as defined in [2].

Materials

1. The almost tangent structure $J : TTM_0 \longrightarrow TTM_0$, defined for each $v \in T_uTM_o$ by
$$J(v) := \frac{d}{dt}|_{t=0}(u + t\pi_*(v)),$$
maps T_uTM_o to $(\mathcal{V}TM_0)_u$.
2. Poincare-Cartan forms $\Theta_L := dL \circ J$ and $\omega_L := -d\Theta_L$, where $L := \frac{1}{2}F^2$.
3. The energy function H by $H := \tilde{1}_{TM_0}(L) - L$, where $\tilde{1}_{TM_0}(L)$ is the vertical vector field corresponding to 1_{TM_0} called redial vector field.
4. The hamiltonian vector field $X_H \in \chi(TM_0)$, defined by
$$\omega_L(X_H, Y) = Y(H), \ \forall Y \in \chi(TM_0)$$
5. For the regular curve $\alpha : I \subseteq \mathbb{R} \longrightarrow M$, length and energy functions are defined by
$$\mathcal{L}(\alpha) = \int_0^1 F(\dot{\alpha}(t))dt, \qquad \mathcal{E}(\alpha) = \int_0^1 L(\dot{\alpha}(t))dt.$$
6. The bundle isomorphism
$$\mathcal{J} : \pi_M^*(TM) \longrightarrow \mathcal{V}TM_0$$
$$(u, v) \longmapsto \frac{d}{dt}|_{t=0}(u + tv)$$
7. The projection map $\mathrm{k} : TTM_0 \longrightarrow TTM_0$ on $\mathcal{V}TM_0$ along $\mathcal{H}TM_0$ called connection form
8. The vector bundle map $k : TTM_0 \longrightarrow TM$ called connection map, defined by
$$\mathrm{k}(v \in T_uTM_0) = \mathcal{J}(u, k(v)).$$

Results

1. Hamiltonian formalism on TM_0

* (TM_0, ω_L) is a symplectic manifold and (TM_0, ω_L, H) a hamiltonian formalism.
* X_H is a restricted spray and its local expression is
$$X_H = \dot{x}^i \frac{\partial}{\partial \tilde{x}^i} - 2G^i \frac{\partial}{\partial \dot{x}^i},$$
where
$$G^i = \frac{1}{4} g^{ij} \left(\dot{x}^k \frac{\partial^2 F^2}{\partial \dot{x}^j \partial \tilde{x}^k} - \frac{\partial F^2}{\partial \tilde{x}^j} \right), \tag{2}$$
* regular curve α is an integral curve of X_H if and only if it is an extremal curve of $\mathcal{L}(\alpha)$. Moreover, if α has constant velocity, then it is an integral curve of X_H if and only if it is an extremal curve of energy function.
* There is a positive homogeneous connection of degree 1, according to which X_H is a horizontal vector field. Related connection map acts on local vector fields by
$$k(\frac{\partial}{\partial \tilde{x}^i}) = \frac{\partial}{\partial x^i} \circ \pi, \qquad k(\frac{\partial}{\partial \dot{x}^i}) = G_i^j \frac{\partial}{\partial x^j},$$
where $G_j^i := \dfrac{\partial G^i}{\partial \dot{x}^j}$.
* The local expression of X_H is $X_H = \dot{x}^j \left(\frac{\partial}{\partial \tilde{x}^j} - G_j^i \frac{\partial}{\partial \dot{x}^i} \right)$.

2. Hamiltonian formalism on $TM_0 \times_f TN_0$

Let (M, F_1) and (N, F_2) be two Finsler manifolds with respective Finsler metrics $\dot{g}$ and $\ddot{g}$ and f be a smooth real function defined on M. The function
$$F(u_1, u_2) := \sqrt{F_1^2(u_1) + f^2(\tau_M(u_1))F_2^2(u_2)}\,,\ (u_1, u_2) \in TM \times TN.$$
is used to define a Hamiltonian structure on $TM_0 \times TN_0$ by $L := \frac{1}{2}F^2$. For $u \in TM_0 \times TN_0$, $v_1 = p_{1*}(v) \in T_{p_1(u)}TM_0$ and $v_2 = p_{2*}(v) \in T_{p_2(u)}TN_0$
* $J(v) = (J_1(v_1), J_2(v_2))$.
* $\Theta_L(v) = \Theta_{L_1}(v_1) + f^2\Theta_{L_2}(v_2)$.
* $\omega_L(v, w) = \omega_{L_1}(v_1, w_1) + f^2\omega_{L_2}(v_2, w_2) - df^2 \wedge \Theta_{L_2}(v_1, w_2) - df^2 \wedge \Theta_{L_2}(v_2, w_1)$

Proposition: Let $X_H = X_1 + X_2$ be the hamiltonian vector field related to hamiltonian formalism $(TM_0 \times TN_0, \omega_L, H)$, then
i) $X_1 = X_{H_1} + Y$, where Y is a section of $\mathcal{V}TM_0$ with local expression $Y = L_2(f^2)^i \frac{\partial}{\partial \dot{x}^i}$.
ii) $X_2 = X_{H_2} + Z$, where Z is a section of $\mathcal{V}TN_0$ with local expression $Z = -d(\ln f^2)\dot{\xi}^\mu \frac{\partial}{\partial \dot{\xi}^\mu}$

Cartan connection and conformal gradient fields

Materials

1. The bundle maps $\varrho, \mu : TTM_0 \longrightarrow \pi^*TM$, defined for each $v \in T_uTM_0$ by
★ $\varrho(v) = (u, \pi_{*u}(v))$, which is a bundle isomorphism from $\mathcal{H}TM_0$ to π^*TM ,
★ $\mu(v) := \nabla_v l$, which is a bundle isomorphism from $\mathcal{V}TM_0$ to π^*TM for regular connections
2. Torsion and curvature , defined for each $\tilde{X}, \tilde{Y} \in \chi(TM_0)$ and $Z \in \Gamma(\pi^*TM)$ by
$$\tau(\tilde{X}, \tilde{Y}) = \nabla_{\tilde{X}}\varrho(\tilde{Y}) - \nabla_{\tilde{Y}}\varrho(\tilde{X}) - \varrho[\tilde{X}, \tilde{Y}], \quad \Omega(\tilde{X}, \tilde{Y})Z = \nabla_{\tilde{X}}\nabla_{\tilde{Y}}Z - \nabla_{\tilde{Y}}\nabla_{\tilde{X}}Z - \nabla_{[\tilde{X}, \tilde{Y}]}Z$$
3. Cartan connection, which is the unique regular metric compatible connection with following properties
$$\tau(\mathcal{H}\tilde{X}, \mathcal{H}\tilde{Y}) = 0, \qquad g(\tau(\mathcal{V}\tilde{X}, \mathcal{H}\tilde{Y}), \varrho(\tilde{Z})) = g(\tau(\mathcal{V}\tilde{X}, \mathcal{H}\tilde{Z}), \varrho(\tilde{Y})),$$
where $\mathcal{V}\tilde{X} = \mathrm{k}(\tilde{X})$ (resp. $\mathcal{H}\tilde{X} = \tilde{X} - \mathrm{k}(\tilde{X})$) is vertical (resp. horizontal) component of $\tilde{X}$.
4. The vector field $\hat{V} \in \chi(TM_0)$, with local flow equal to $(\phi_t)_*$, is called the complete lift of the nowhere zero vector filed $V \in \chi(M)$ to TM, cf. [9]. It can be shown that $\varrho(\hat{V}) = V$ and $\mu(\hat{V}) = \nabla_{\hat{l}}V$.
5. Let $f \in C^\infty(M)$. At each point $p \in M$ that $df_p \neq 0$, $\mathrm{grad}f(p)$ is a vector on T_pM with the following property, cf. [7],
$$g_{\mathrm{grad}f(p)}(u, \mathrm{grad}f(p)) = df_p(X), \ \forall u \in T_pM.$$

Results on Conformal gradient fields

* For $(0, k)$-tensor W in Finsler sense,
$$(\mathcal{L}_{\hat{V}}W)(X_1, X_2, ..., X_k) = (\nabla_{\hat{V}}W)(X_1, X_2, ..., X_k) + \Sigma_i W(X_1, ..., X_{i-1}, \nabla_{\mathcal{H}\tilde{X}_i}V, X_{i+1}, ..., X_k)$$
$$+ \Sigma_i W\left(X_1, ..., X_{i-1}, \tau(\mathcal{V}\hat{V}, \mathcal{H}\tilde{X}_i), X_{i+1}, ..., X_k\right),$$
where $\mathcal{H}\tilde{X}_i$ is the unique horizontal vector field that maps to X_i under ϱ.
* Let for $f \in C^\infty(M)$, $\mathrm{grad}f$ be a conformal vector field on M. Then the integral curves of $\mathrm{grad}f$ are geodesics of the Finsler structure.
* Let $\nabla_{\mathcal{H}\tilde{X}}\mathrm{grad}f = 0$, for each $\tilde{X} \in \chi(TM_0)$. Then in a neighborhood of each ordinary point of $\mathrm{grad}f$, level surfaces of f are locally isometric.
* Let $\mathrm{grad}f$ be a conformal vector field on M and p an ordinary point of $\mathrm{grad}f$. Then the norm of $\mathrm{grad}f$ is constant on the $f-$hypersurface that contains p.

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