Pearl Complex With Local Coefficients: The Chiang Lagrangian and $\mathbb{R}P^3$

Momchil Konstantinov

University College London, London School of Geometry and Number Theory

The Pearl Complex with Local Coefficients

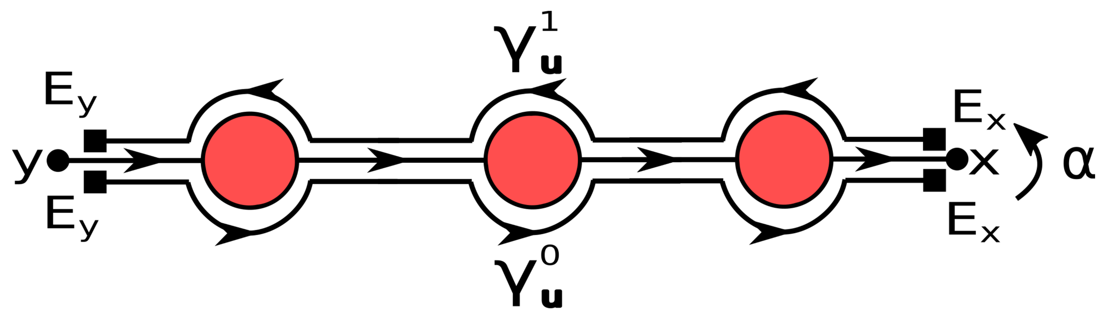
Setup

Let L be a compact monotone Lagrangian, equipped with a local system $E \to L$ of vector spaces over $\mathbb{F}_2$. We twist the coefficients of the Biran-Cornea pearl complex by this local system. For a Morse function $f \colon L \to \mathbb{R}$ one defines:

$$C_f^*(L; \mathrm{End}(E)) = \bigoplus_{x \in Crit(f)} \mathrm{End}_{\mathbb{F}_2}(E_x)$$

The Differential

A differential on $C_f^*(L; \mathrm{End}(E))$ is defined using rigid pearly trajectories for parallel transport.



$$d(\alpha) = \sum_{\substack{y \in Crit(f) \\ |y| = |x| - kN_L + 1}} \sum_{\mathbf{u} \in \mathcal{P}(y,x,kN_L)} P_{\gamma_{\mathbf{u}}^1} \circ \alpha \circ P_{\gamma_{\mathbf{u}}^0}.$$

Whenever $d^2 = 0$ we define:

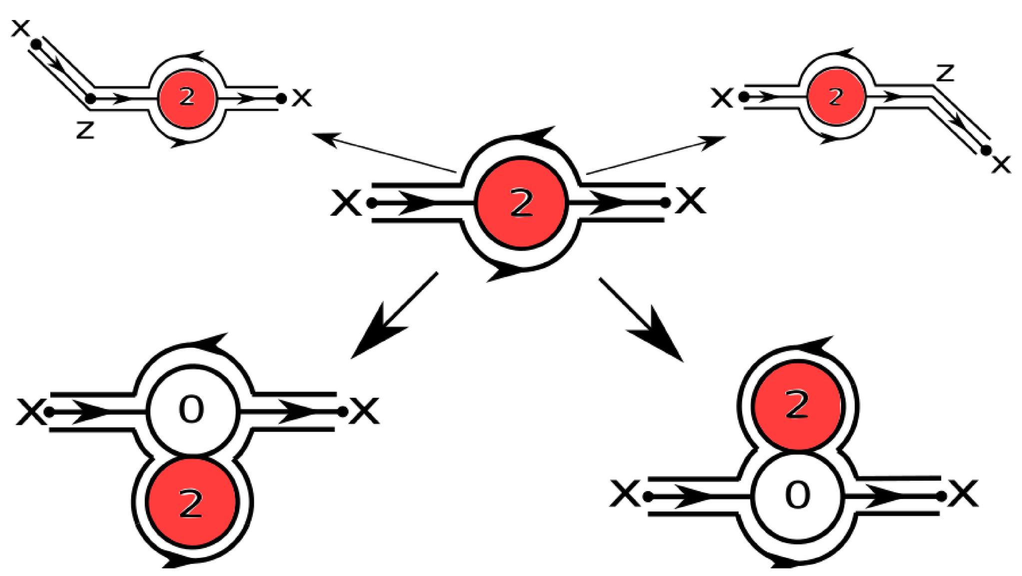
$$\boxed{HF((L,E),(L,E)) \coloneqq H(C_f^*(L; \mathrm{End}(E)), d)}$$

The Obstruction

One does not necessarily get $d^2 = 0$. Instead:

$$d^2(\alpha) = \alpha \circ m_0(E)(x) + m_0(E)(x) \circ \alpha,$$

where $m_0(E)(x) \coloneqq \sum_{u \in \mathcal{M}_{0,1}(x,2,L)} P_{\partial u}$.



Contact: momchil.konstantinov.14@ucl.ac.uk

The Chiang Lagrangian

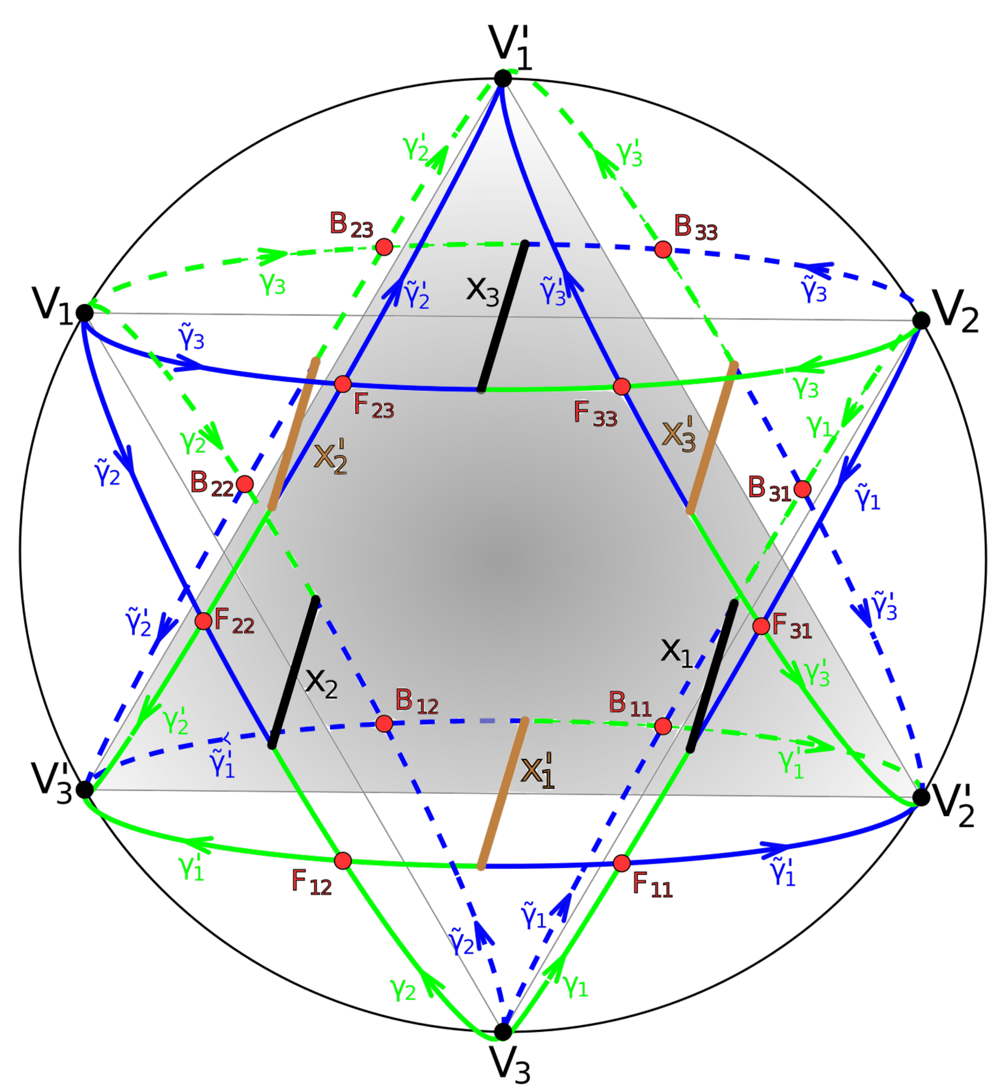
The Chiang Lagrangian L_Δ is a Lagrangian submanifold of $\mathbb{C}P^3 = Sym^3(\mathbb{C}P^1)$. It is the orbit of the polynomial $x^3 - y^3$ under the natural $SU(2)$ action. The stabiliser is the binary dihedral group of order 12, i.e. $\pi_1(L_\Delta) = \langle a, b \mid a^6 = 1,\ b^2 = a^3,\ ab = ba^{-1} \rangle$, where $a = \exp(\mathbf{i}\pi/3)$ and $b = \exp(\mathbf{k}\pi/2)$. The Chiang Lagrangian intersects $\mathbb{R}P^3$ cleanly in two disjoint circles. Our main result is the following:

> The Chiang Lagrangian and $\mathbb{R}P^3$ are not Hamiltonianly displaceable.

The Proof

We use the rank 2 local system $E \to L_\Delta$ obtained from the representation $\pi_1(L_\Delta) \to D_3 \cong GL(2, \mathbb{F}_2)$, in order to make (L_Δ, E) an essential object in an extension of the Fukaya category $\mathcal{F}_{\mathbb{F}_2}(\mathbb{C}P^3)$. This category is split-generated by $\mathbb{R}P^3$ by a result of Tonkonog [1], which yields the conclusion. Checking that (L_Δ, E) is essential is an explicit calculation using the pearl complex.

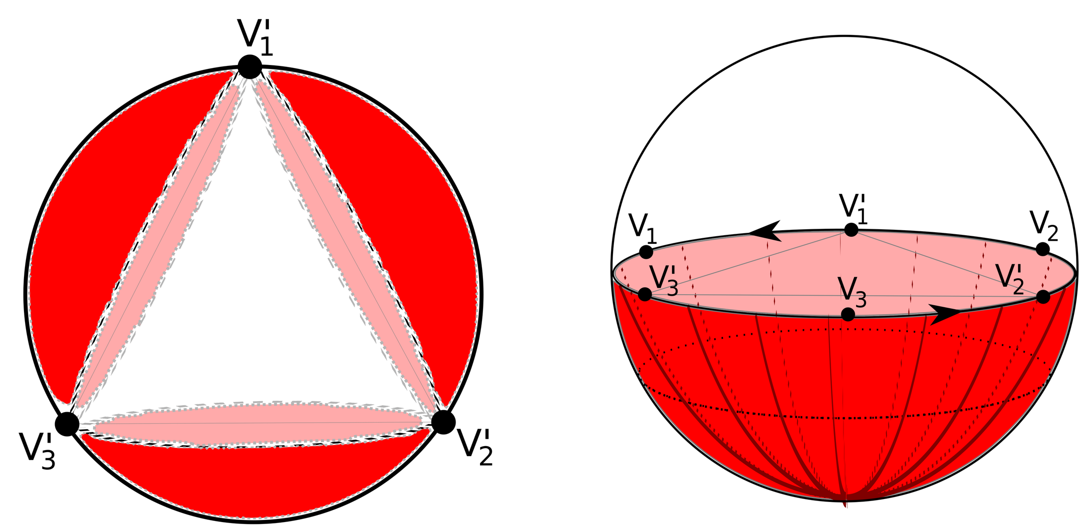
A Morse Function



The minimum is at $m' \coloneqq \triangle V_1' V_2' V_3'$ and the maximum at $m \coloneqq \triangle V_1 V_2 V_3$. Critical points of index one $\{x_i'\}_{1 \le i \le 3}$ and index two $\{x_i\}_{1 \le i \le 3}$ are triangles with one side along the respective segment.

Axial Discs

In [2] Evans and Lekili show that all holomorphic discs counted by the differential come from the global $SL(2, \mathbb{C})$−action on $\mathbb{C}P^3$.



One reads off $m_0(E)(m') = (1 + a^2 + a^4)b = 0$, so $HF((L_\Delta, E), (L_\Delta, E))$ is well-defined.

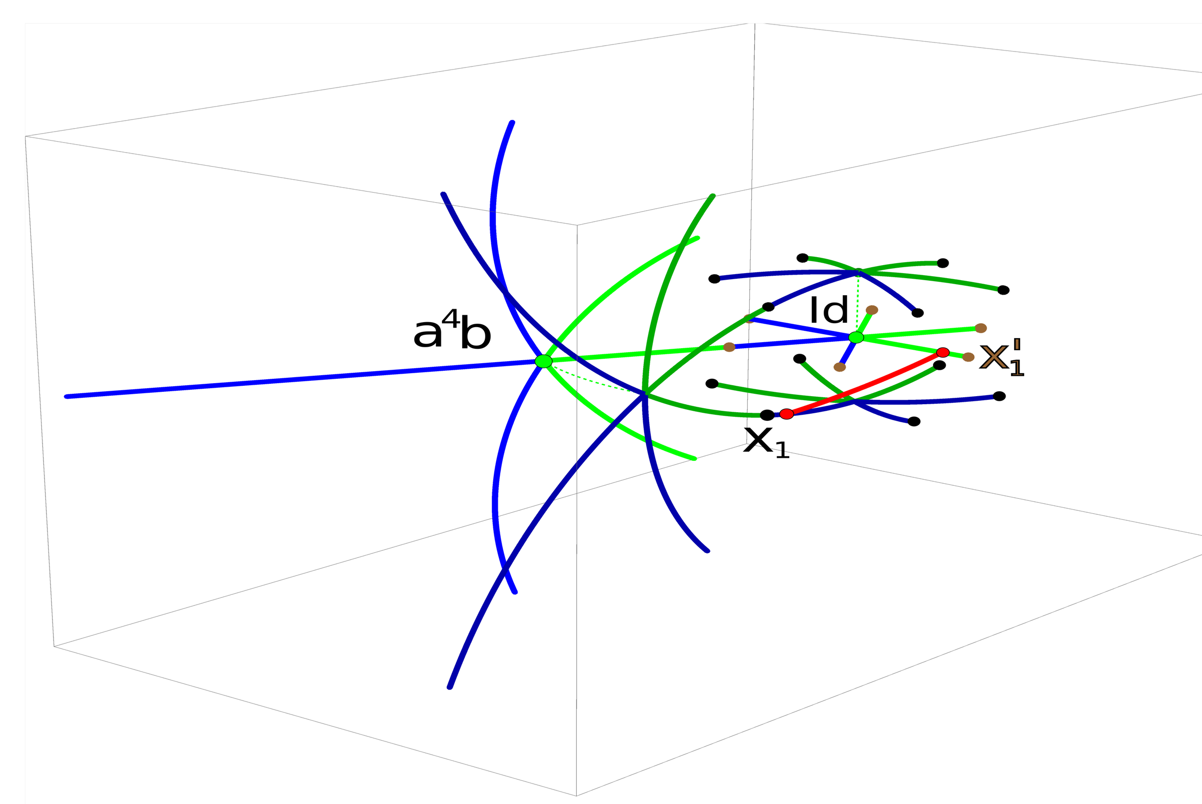
The rigid pearly trajectories entering d are:

- three Maslov 2 trajectories through m' and three through m
- twelve Maslov 2 trajectories containing a disc with axis F_{ij}, B_{ij} and connecting x_i' to x_j.
- two Maslov 4 trajectories connecting m' to m

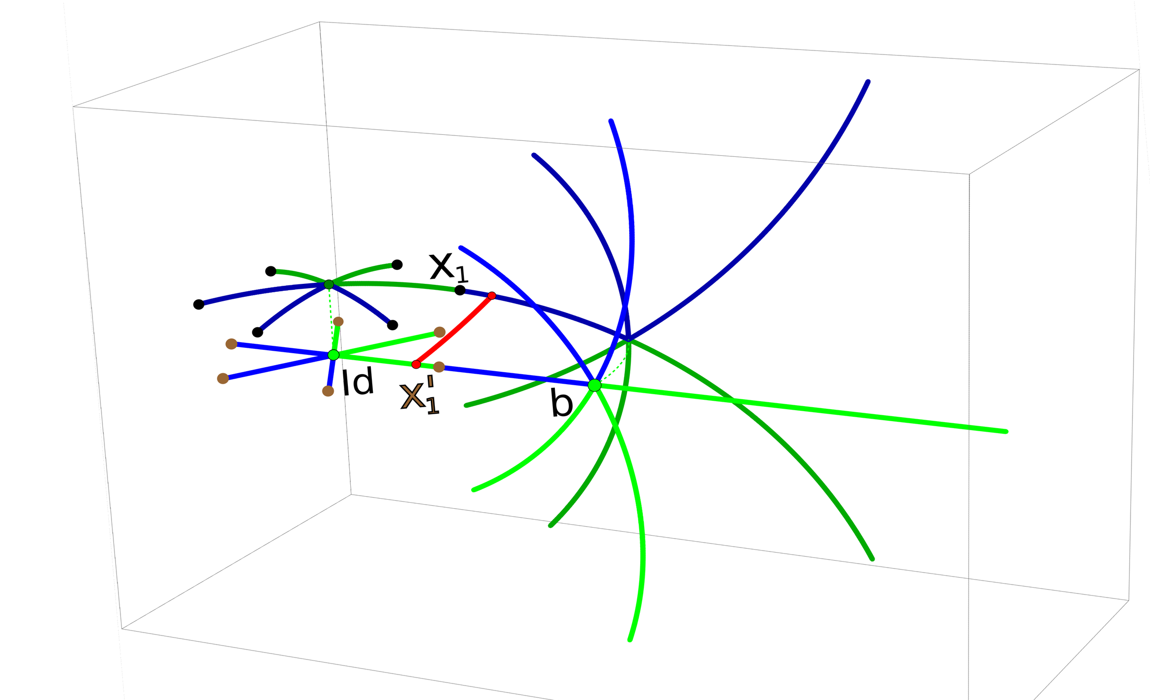
Lifting Paths

Parallel transport is calculated by lifting paths to the universal cover $S^3 = SU(2) \to L_\Delta$.

A lift of $\gamma_{B_{11}}^0$, showing $P_{\gamma_{B_{11}}^0} = a^4 b$:



A lift of $\gamma_{B_{11}}^1$, showing $P_{\gamma_{B_{11}}^0} = \mathrm{Id}$:



Applying this procedure to all trajectories yields the full differential and one directly computes

$$HF((L_\Delta, E), (L_\Delta, E)) \cong (\mathbb{F}_2)^4. \quad \square$$

Acknowledgements

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References

[1] Dmitry Tonkonog.
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[2] Jonathan David Evans and Yankı Lekili.
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