Dehn twists exact sequences through Lagrangian cobordism

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Abstract

We give a new Lagrangian surgery construction after Lalonde-Sikorav and Polterovich's well-known construction, and combined this with Biran-Cornea's Lagrangian cobordism formalism. Together we build a powerful framework which not only recovers many known long exact sequences (Seidel's exact sequence including the fixed point version, Wehrheim-Woodward's family version, and the connecting map in Fukaya-Oh-Ohta-Ono surgery triangle) in symplectic geometry in a uniform way, but also yields an answer to the long-term open conjecture due to Huybrechts and Thomas. This also involved a new observation which relates projective twists with surgeries. Moreover, we established an immersed cobordism framework for computation of connecting maps, which should be of independent interests.

Seidel's Dehn twist exact triangle

Given a (symplectic) Lefschetz fibration $f: M \to \mathbb{C}$, the monodromy around a critical value is given by symplectic Dehn twist τ_S along the vanishing cycle S.

The most fundamental and influential exact triangle in Lagrangian Floer theory is

Theorem 1. *([7][8]) For a Lagrangian spherical brane S and exact Lagrangian branes L_0, L_1 in a Liouville domain (M, ω), there is a long exact sequence*

$$\cdots \to HF(S, L_1) \otimes HF(L_0, S) \to HF(L_0, L_1) \to HF(L_0, \tau_S(L_1)) \to \dots$$

In derived Fukaya category, there is an exact triangle

$$\begin{array}{ccc} CF(S, L_1) \otimes S & \longrightarrow & L_1 \\ {\scriptstyle [1]}\uparrow & \swarrow & \\ \tau_S(L_1) & & \end{array}$$

which describes the auto-equivalence induced by τ_S.

Spherical object and spherical twist are defined on the algebraic geometric side which mimic the effect of Dehn twist on the derived Fukaya category.

Huybrechts-Thomas's conjecture

Apart from spheres, Dehn twist can also be defined for other Lagrangian submanifolds such as real, complex and quaternionic projective spaces. For a smooth projective variety X, an object $\mathcal{E} \in D^b(X)$ is called a $\mathbb{P}^n$ object if $\mathcal{E} \otimes \omega_X = \mathcal{E}$ and $Ext^*(\mathcal{E}, \mathcal{E})$ is isomorphic to $H(\mathbb{P}^n, \mathbb{K})$ as graded ring.

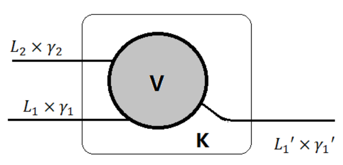
Huybrechts and Thomas defined an autoequivalence of $D^b(X)$ by twisting along $\mathbb{P}^n$ objects ([5]). This is the Fourier-Mukai functor with Fourier-Mukai kernel being the following iterated mapping cone in $D^b(X \times X)$

$$Cone(Cone(\check{\mathcal{E}} \boxtimes \mathcal{E}[-2] \to \check{\mathcal{E}} \boxtimes \mathcal{E}) \to \mathcal{O}_\Delta) \quad (1)$$

This auto-equivalence can be translated into A_∞ categories ([4]). It is conjectural that the auto-equivalences on the derived Fukaya category induced by Dehn twists along Lagrangian projective spaces are twists along $\mathbb{P}^n$ objects.

Biran-Cornea's Lagrangian cobordism

A Lagrangian cobordism V from L_1, L_2 to L_1' is a Lagrangian in $M \times \mathbb{C}$ such that it is the union of product Lagrangians $L_1 \times \gamma_1$, $L_2 \times \gamma_2$ and $L_1' \times \gamma_1'$ in the clockwise order outside $M \times K$ for some compact set $K \subset \mathbb{C}$ and some properly embedded disjoint simple curves $\gamma_1, \gamma_2, \gamma_1'$.



Theorem 2. *([1][2]) In the exact or monotone setting, if there is a Lagrangian cobordism V from $L_1[1], L_2$ to L_1' then there is an exact triangle*

$$\begin{array}{ccc} L_1 & \longrightarrow & L_2 \\ {\scriptstyle [1]}\uparrow & \swarrow & \\ L_1' & & \end{array}$$

in the drived Fukaya category.

The result works analogously when the Lagrangian cobordism has more ends. One important application is to recover the following surgery exact triangle.

Example 3. ([1],[3]) When L_0 intersects L_1 transversally at a point p of index 0, there is a Lagrangian cobordism from $L_0[1], L_1$ to $L_0[1]\#_p L_1$. As a result, we have $Cone(L_0 \to L_1) \simeq L_0[1]\#_p L_1$.

Mau-Wehrheim-Woodward's functor and Lagrangian correspondence

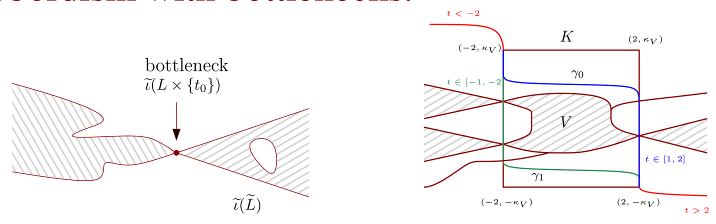
A Lagrangian correspondence $L_{0,1}$ is a Lagrangian in $(M_0 \times M_1, -\omega_0 \oplus \omega_1)$. For a sequence of symplectic manifolds $M_0, \dots, M_{r+1}$ such that $M_0 = M_{r+1}$, a cyclic generalized Lagrangian correspondence is a sequence of Lagrangian correspondences $(L_{0,1}, \dots, L_{r,r+1})$. In the exact/monotone setting, one can define quilted Floer cohomology ([12]) for a cyclic generalized Lagrangian correspondence $HF(L_{0,1}, \dots, L_{r,r+1})$. The simplest examples are

- $HF(L_1 \times L_2, \Delta_M) = HF(L_1, L_2)$
- $HF(L_1 \times L_2, Graph(\phi^{-1})) = HF(L_1, \phi(L_2))$
- $HF(L_1 \times L_2, K_1 \times K_2) = HF(L_1, K_1) \otimes HF(K_2, L_2)$

More generally, a (generalized) Lagrangian correspondence $L_{0,1}$ should induce an A_∞ functor between certain Fukaya categories ([11], [9]).

Dehn twist via immersed Lagrangian cobordism

Let $P = \mathbb{CP}^{\frac{n}{2}}$ and L be Lagrangian branes. If $P \pitchfork L = \{p\}$, then $\tau_P(L)$ is diffeomorphic to $P\#_D P\#_p L = S_{\looparrowright}\#_p L$, where D is the divisor opposite to p and $S_{\looparrowright}$ is an immersed sphere with a single immersed point. We adapt Lagrangian cobordism in this immersed setting. To guarantee compactness, we require the ends of the cobordism are modelled on pinched Lagrangians and call it an immersed Lagrangian cobordism with bottlenecks.



By comparing $HF(V, N \times \gamma_0)$ and $HF(V, N \times \gamma_1)$, we can show that

Theorem 4. *[6] Let V be a graded exact immersed Lagrangian cobordism with bottlenecks from $L_1[1], L_2$ to L_3 such that all immersed points with index ≤ 2 have negative energy. Then for any cleanly immersed graded exact Lagrangian N with immersed points satisfying the same assumption, we have*

$$Cone(CF(N, L_1) \to CF(N, L_2)) \simeq CF(N, L_3)$$

Remark 5. There are counter-examples if the assumption on immersed points is violated.

By generalizing Lalonde-Sikorav and Polterovich's construction to cleanly intersecting situation, we can construct an immersed Lagrangian cobordism with bottlenecks from $P[-1], P$ to $S_{\looparrowright}$ and from $S_{\looparrowright}[1], L$ to $\tau_P(L)$, satisfying the assumption. As a result, we have

Theorem 6. *[6] For any cleanly immersed graded exact Lagrangian N with immersed points satisfying the assumption, we have*

$$Cone(CF(N, P)[-2] \to CF(N, P)) \simeq CF(N, S_{\looparrowright})$$

$$Cone(CF(N, S_{\looparrowright}) \to CF(N, L)) \simeq CF(N, \tau_P(L))$$

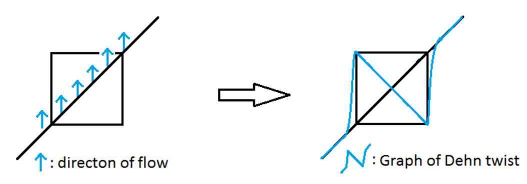
One should compare the formal and the latter equations with the the first and the second mapping cones in equation 1.

Dehn twist via Lagrangian correspondence

When P and L intersect more than one point, we need a better perspective to understand the projective Dehn twist. Instead of realizing $\tau_P(L)$ as a mapping cone, we want to get an exact triangle on the functor level. We obtain the following surgery identities in $(M \times M, \omega \oplus -\omega)$

- $(S \times S)[1]\#_{\Delta_S}\Delta = Graph(\tau_S^{-1})$
- $(P \times P)\#_{\mathcal{D}}(P \times P)[1]\#_{\Delta_P}\Delta = Graph(\tau_P^{-1})$

and construct the corresponding Lagrangian cobordisms by resolving the clean intersections along one direction.



The first surgery identity corresponds to spherical Dehn twist and the second one corresponds to the complex projective space Dehn twist. Similar statements are true for Dehn twist along real/quaternionic projective spaces. More generally, one can define fiberwise Dehn twist for coisotropic submanifolds C if the fibers are spheres or projective spaces ([10]). We can perform the one directional clean surgery fiberwisely and obtain the corresponding surgery identities for fiberwise Dehn twist.

Theorem 7. *([6]) In the monotone setting, we have a functor level cone*

$$Cone(Cone(P \times P[-2] \to P \times P) \to Id) \simeq \tau_P$$

where $P \times P$ is a bimodule viewed as a functor. Analogous results for spherical Dehn twist and spherical/projective fibered coisotropic Dehn twist can be obtained using the same method.

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