An A_∞ algebra for Legendrians from Generating Families

Ziva Myer
Bryn Mawr College

Legendrian Submanifolds

Given a manifold M^n, the **1-jet space of M**, $J^1(M) = T^*M \times \mathbb{R}$ has a contact structure $\xi = \ker(dz - \sum y_i dx_i)$.

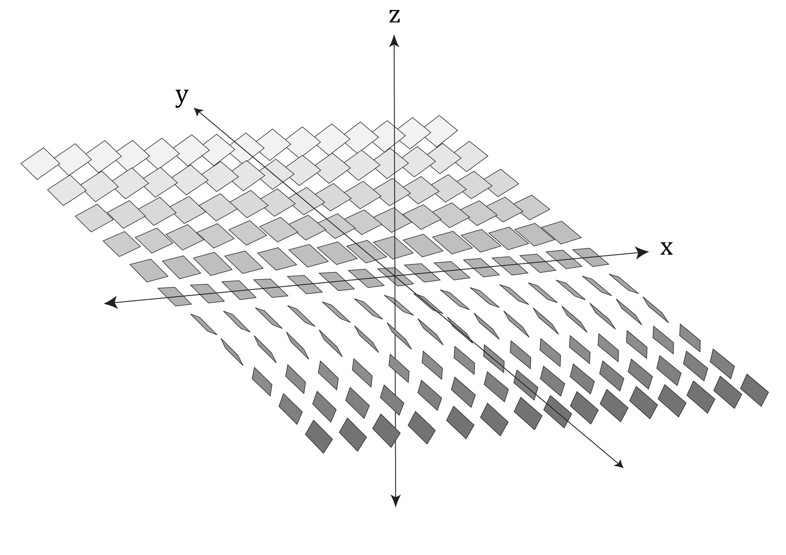
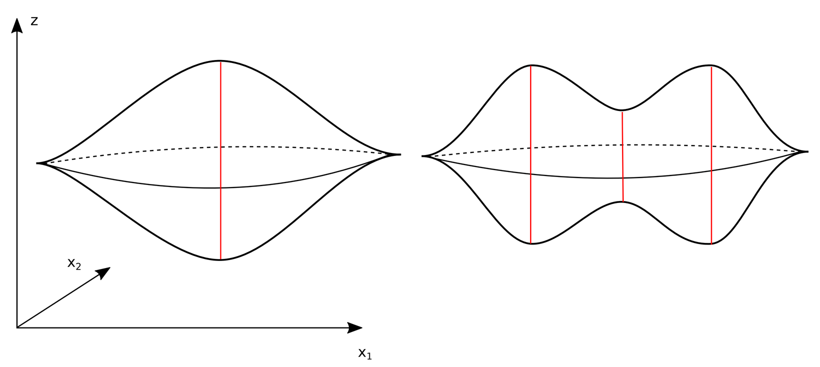


Figure 1: The **standard contact structure** on $\mathbb{R}^3 = J^1(\mathbb{R})$ is given by $ker(dz - ydx)$.

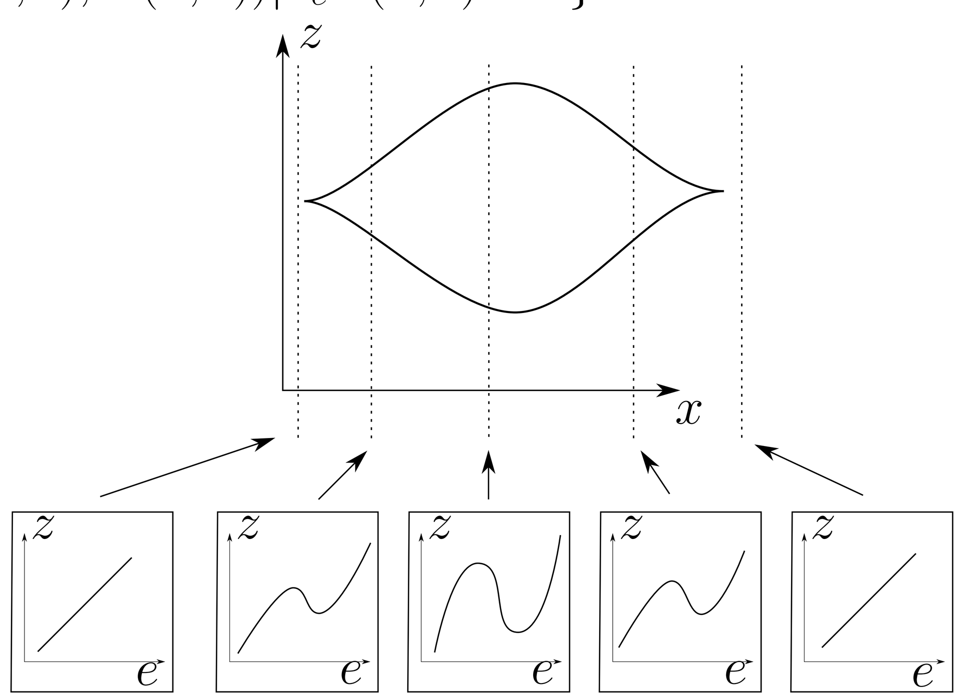
An n-dimensional submanifold $\Lambda \subset J^1(M)$ is **Legendrian** if its tangent space $T_p\Lambda \subset \xi$ for all $p \in \Lambda$.



The Reeb vector field $\frac{\partial}{\partial z}$ determines Reeb Chords, which are important features of Λ. Much work in Contact Geometry involves defining algebraic invariants for Legendrians from Reeb chords and our **goal** is to extend such invariants.

Generating Families

If $F : M \times \mathbb{R}^N \longrightarrow \mathbb{R}$ is a generating family for $\Lambda \subset (J^1(M), \xi)$ then $\Lambda = \{(x, \partial_x F(x,e), F(x,e)) | \partial_e F(x,e) = 0\}$.



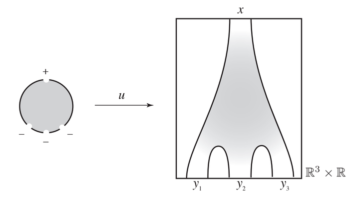
The generating family gives rise to the **difference function** $w : M \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$

$$w(x, e_1, e_2) = F(x, e_1) - F(x, e_2)$$

whose positive-valued critical points form a graded vector space $C(F)$ over $\mathbb{Z}_2$ and are in bijective correspondence with Reeb chords of Λ.

Motivation

There are parallels between two different techniques for constructing algebraic invariants for Legendrians from Reeb chords:

Pseudoholomorphic Curves	**Generating Families**
DGA $(\mathcal{A}, \partial)$, $\mathcal{A} = \oplus_{k=0}^{\infty} A^{\otimes k}$  ∂ counts:	?
Augmentation $\epsilon : \mathcal{A} \longrightarrow \mathbb{Z}_2$ $\partial^\epsilon : A \longrightarrow A$	Generating Family F Positive gradient flow from w.
$\{LCH^*(\epsilon)\}_\epsilon$	$\{GH^*(F)\}_F = \{H^*_{\text{Morse}}(C(F))\}_F$ [Traynor, Fuchs-Rutherford]
A_∞-algebra $m_k : A^{\otimes k} \longrightarrow A$ [Etnyre-Sabloff-et al.]	$\exists\, m_k : C(F)^{\otimes k} \longrightarrow C(F)$? Yes! [My thesis work]

Moduli Spaces of Gradient Flow Trees

A space $\mathcal{M}(p_1, p_2, \cdots, p_k; p_0)$ of gradient flow trees is built using **extended difference functions**. $w_{i,j;k+1} : M \times \mathbb{R}^{(k+1)N} \longrightarrow \mathbb{R}$

$$w_{i,j;k+1}(x, e_1, \cdots, e_{k+1}) = F(x, e_i) - F(x, e_j) + \sum_{\ell=1}^{i-1} e_\ell^2 - \sum_{\ell=i+1}^{j-1} e_\ell^2 + \sum_{\ell=j+1}^{k+1} e_\ell^2,$$

Positive-valued critical points of $w_{i,j;k+1} \iff$ Reeb chords.

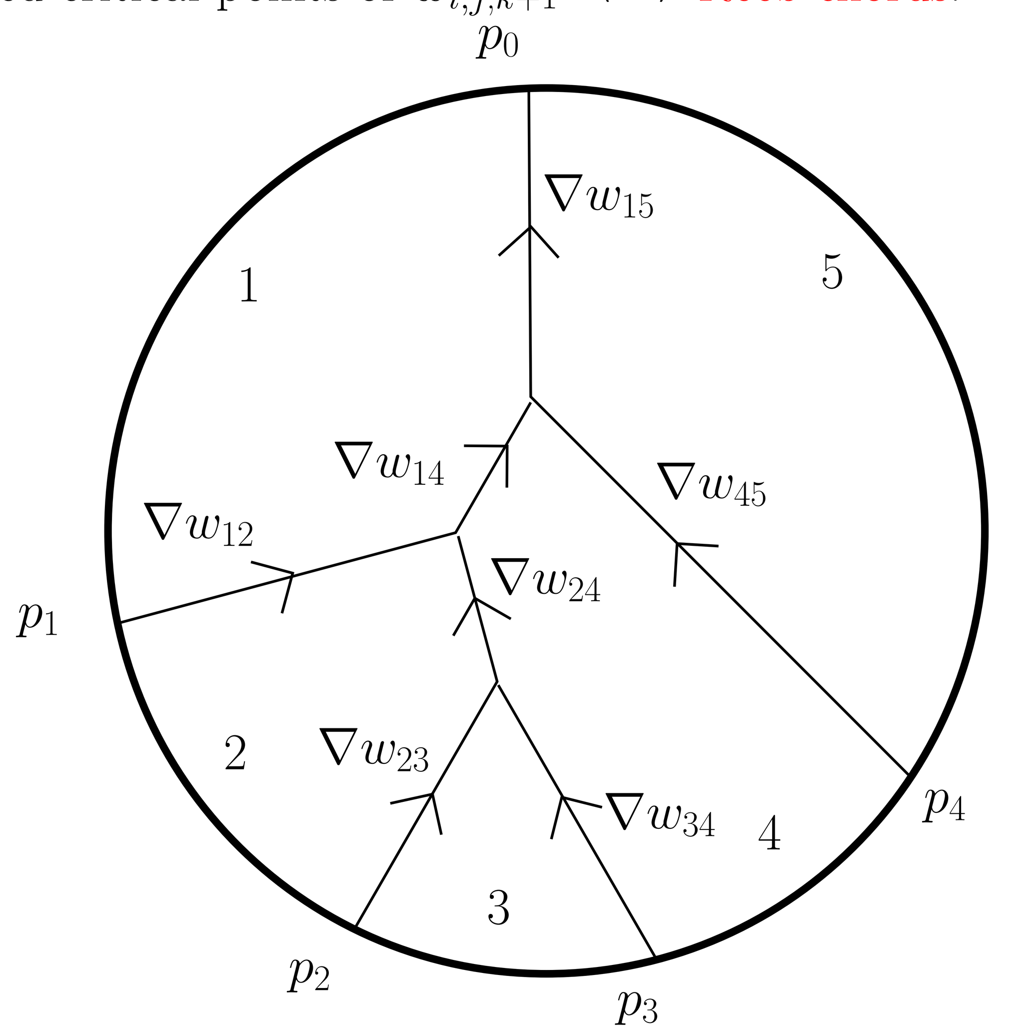


Figure 2: An element in $\mathcal{M}(p_1, p_2, p_3, p_4; p_0)$.

$\mathcal{M}_F(p_1, \ldots, p_k; p_0)$ is a manifold of dimension

$$|p_0| - \left(\sum_{i=1}^{k} |p_i| + (k-2) \right).$$

The unbroken trees lives in a larger metrizable space of trees that has a manifold with corners structure.

Cochain Product Maps

On generators define

$$m_k : C^{i_1}(F) \otimes \cdots \otimes C^{i_k}(F) \longrightarrow C^{j'}(F)$$
$$m_k(p_1 \otimes \cdots \otimes p_k) = \sum (\#_{\mathbb{Z}_2} \mathcal{M}(p_1, p_2, \cdots, p_k; p_0)) \cdot p_0$$

where the sum is taken over $p_0 \in C^{j'}(F)$ where $j' = \sum_{\ell=1}^{k} i_\ell + k - 2$. Studying the compactification of a 1-dimensional $\mathcal{M}(p_1, p_2, \cdots, p_k; p_0)$ shows that the m_k maps satisfy the A_∞ **relations**:

$$\sum_{i+j+\ell=k} m_{i+1+\ell} \circ (1^{\otimes i} \otimes m_j \otimes 1^{\otimes \ell}) = 0.$$

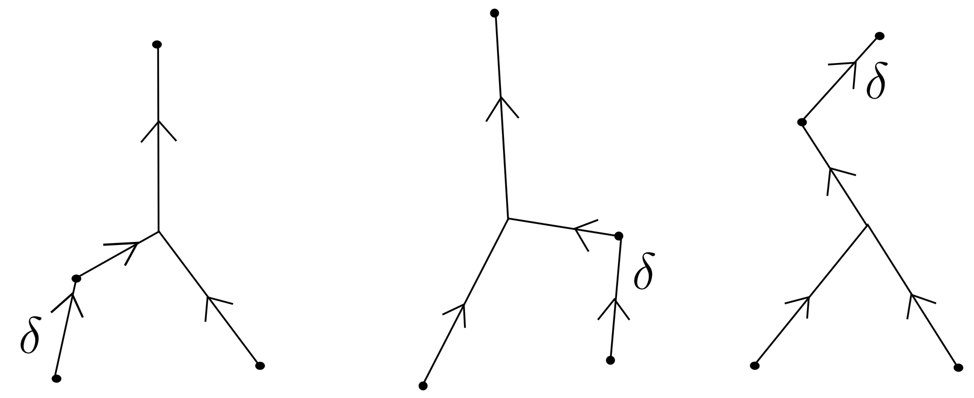
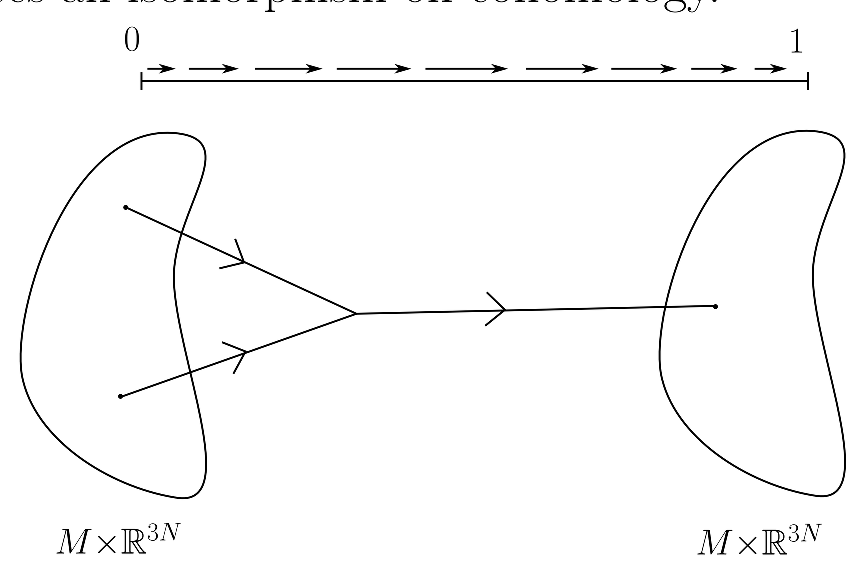


Figure 3: In particular, m_2 is a cochain map.

Given a Legendrian isotopy from Λ_0 to Λ_1, we build spaces of trees in $(M \times \mathbb{R}^{(k+1)N}) \times I$. Studying the compactification of 1-dimensional spaces of such trees yields an A_∞ quasi-isomorphism, maps $\phi_k : C(F_0)^{\otimes k} \longrightarrow C(F_1)$ that satisfy

$$\sum_{i+j+l=k} \phi_{i+1+l} \circ (1^{\otimes i} \otimes m_j \otimes 1^{\otimes l}) = \sum_{\substack{1 \le r \le n \\ i_1 + \cdots + i_r = n}} n_r \circ (\phi_{i_1} \otimes \cdots \otimes \phi_{i_r}).$$

where ϕ_1 induces an isomorphism on cohomology.



The following theorem will give an A_∞ algebra structure on $GH^*(F)$ after applying Kadeishvili's Minimal Model Theorem.

Main Theorem [in progress]

- $(C(F), \mathbf{m} = \{m_k\}_{k=1}^{\infty})$ is an A_∞ algebra,
- $H^*(C(F), m_1) = GH^*(F)$, and
- If Λ is Legendrian isotopic to $\widehat{\Lambda}$, $(C(F), \mathbf{m})$ is A_∞ quasi-isomorphic to $(C(\widehat{F}), \widehat{\mathbf{m}})$.