Cohomological Hall algebras, semicanonical bases and Donaldson-Thomas invariants for 2-dimensional Calabi-Yau categories

with an appendix by Ben Davison

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Cohomological Hall algebras and semicanonical bases

For any dimension vector $\gamma = (\gamma^i)_{i\in I} \in \mathbb{Z}^I_{\geqslant 0}$ we have the following algebraic varieties:

- the space $\mathbf{M}_{\overline{Q},\gamma}$ of representations of the double quiver $\overline{Q}$ in coordinate spaces $(\mathbb{C}^{\gamma^i})_{i\in I}$;
- the similar space of representations $\mathbf{M}_{\Pi_Q,\gamma}$ of Π_Q;
- the similar space of representations $\mathbf{M}_{\widehat{Q},\gamma}$ of $\widehat{Q}$.

All these spaces of representations are endowed with the action by conjugation of the complex algebraic group $\mathbf{G}_\gamma = \prod_{i\in I} GL(\gamma^i, \mathbb{C})$.

- $Tr(W)_\gamma = \sum_{i\in I, k=1,\dots,(\gamma^i)^2} f_{ik}x_{ik}$: $\mathbf{G}_\gamma$-equivariant function on $\mathbf{M}_{\widehat{Q},\gamma}$, where f_{ik} are functions on $\mathbf{M}_{\overline{Q},\gamma}$, and $\{x_{ik}\}$ is a linear coordinate system on $\mathbb{A}^{\gamma\cdot\gamma}$;
- $\mathbf{M}_{\Pi_Q,\gamma}$ is the common zero of f_{ik}'s;
- $\mathbf{M}_{\Pi_Q,\gamma_1,\gamma_2}$: the space of representations of $\overline{Q}$ in coordinate spaces of dimension $\gamma_1+\gamma_2$ s.t. the standard coordinate subspaces of dimension γ_1 form a subrepresentation, and the restriction of $\rho \in \mathbf{M}_{\Pi_Q,\gamma_1,\gamma_2}$ on the block-diagonal part is an element in $\mathbf{M}_{\Pi_Q,\gamma_1} \times \mathbf{M}_{\Pi_Q,\gamma_2}$;
- $\mathbf{G}_{\gamma_1,\gamma_2} \subset \mathbf{G}_\gamma$: the group consisting of transformations preserving subspaces $(\mathbb{C}^{\gamma_1^i} \subset \mathbb{C}^{\gamma^i})_{i\in I}$ acts on $\mathbf{M}_{\Pi_Q,\gamma_1,\gamma_2}$;
- $\mathbf{M}^{sp}_{\overline{Q},\gamma} \subset \mathbf{M}_{\overline{Q},\gamma}$: the space of seminilpotent representations of dimension γ;
- $\mathbf{M}^{sp}_{\Pi_Q,\gamma} \subset \mathbf{M}^{sp}_{\overline{Q},\gamma}$: the space of seminipotnet representations of Π_Q of dimension γ, a Lagrangian subvariety of $\mathbf{M}_{\overline{Q},\gamma}$;
- $\mathbf{M}^{sp}_{\widehat{Q},\gamma} = \mathbf{M}^{sp}_{\overline{Q},\gamma} \times \mathbb{A}^{\gamma\cdot\gamma}$;
- φ_γ: sheaf of vanishing cycles of $Tr(W)_\gamma$;
- $\chi_Q(\gamma_1,\gamma_2) = -\sum_{i,j\in I} a_{ij}\gamma_1^i\gamma_2^j + \sum_{i\in I}\gamma_1^i\gamma_2^i$: the Euler form on the K_0 group of the category of finite dimensional representations of Q.

Results

Let $\mathcal{H}_\gamma := H^\bullet_{c,\mathbf{G}_\gamma}(\mathbf{M}^{sp}_{\Pi_Q,\gamma},\mathbb{Q})^\vee \otimes \mathbb{T}^{-\chi_Q(\gamma,\gamma)}$, and $\mathcal{H} = \bigoplus_{\gamma\in\mathbb{Z}^I_{\geqslant 0}} \mathcal{H}_\gamma$. Since $H^{\bullet,crit}_{c,\mathbf{G}_\gamma}(\mathbf{M}^{sp}_{\widehat{Q},\gamma}, W_\gamma) := H^\bullet_{c,\mathbf{G}_\gamma}(\mathbf{M}^{sp}_{\widehat{Q},\gamma},\varphi_\gamma) \simeq H^\bullet_{c,\mathbf{G}_\gamma}(\mathbf{M}^{sp}_{\Pi_Q,\gamma},\mathbb{Q}) \otimes \mathbb{T}^{\gamma\cdot\gamma}$, the coproduct on $\oplus_\gamma H^{\bullet,crit}_{c,\mathbf{G}_\gamma}(\mathbf{M}^{sp}_{\widehat{Q},\gamma}, W_\gamma)$ induces associative product on $\mathcal{H}$:

$$
\begin{aligned}
&\mathcal{H}_{\gamma_1}\otimes\mathcal{H}_{\gamma_2} = H^\bullet_{c,\mathbf{G}_{\gamma_1}}(\mathbf{M}^{sp}_{\Pi_Q,\gamma_1},\mathbb{Q})^\vee \otimes \mathbb{T}^{-\chi_Q(\gamma_1,\gamma_1)} \otimes H^\bullet_{c,\mathbf{G}_{\gamma_2}}(\mathbf{M}^{sp}_{\Pi_Q,\gamma_2},\mathbb{Q})^\vee \otimes \mathbb{T}^{-\chi_Q(\gamma_2,\gamma_2)}\\
=\ & H^\bullet_{c,\mathbf{G}_{\gamma_1}}(\mathbf{M}^{sp}_{\Pi_Q,\gamma_1},\mathbb{Q})^\vee \otimes H^\bullet_{c,\mathbf{G}_{\gamma_2}}(\mathbf{M}^{sp}_{\Pi_Q,\gamma_2},\mathbb{Q})^\vee \otimes \mathbb{T}^{-\chi_Q(\gamma_1,\gamma_1)-\chi_Q(\gamma_2,\gamma_2)}\\
\to\ & H^\bullet_{c,\mathbf{G}_{\gamma_1+\gamma_2}}(\mathbf{M}^{sp}_{\Pi_Q,\gamma_1+\gamma_2},\mathbb{Q})^\vee \otimes \mathbb{T}^{-\chi_Q(\gamma_1,\gamma_2)-\chi_Q(\gamma_2,\gamma_1)} \otimes \mathbb{T}^{-\chi_Q(\gamma_1,\gamma_1)-\chi_Q(\gamma_2,\gamma_2)}\\
=\ & H^\bullet_{c,\mathbf{G}_{\gamma_1+\gamma_2}}(\mathbf{M}^{sp}_{\Pi_Q,\gamma_1+\gamma_2},\mathbb{Q})^\vee \otimes \mathbb{T}^{-\chi_Q(\gamma_1+\gamma_2,\gamma_1+\gamma_2)} = \mathcal{H}_{\gamma_1+\gamma_2}.
\end{aligned}
$$

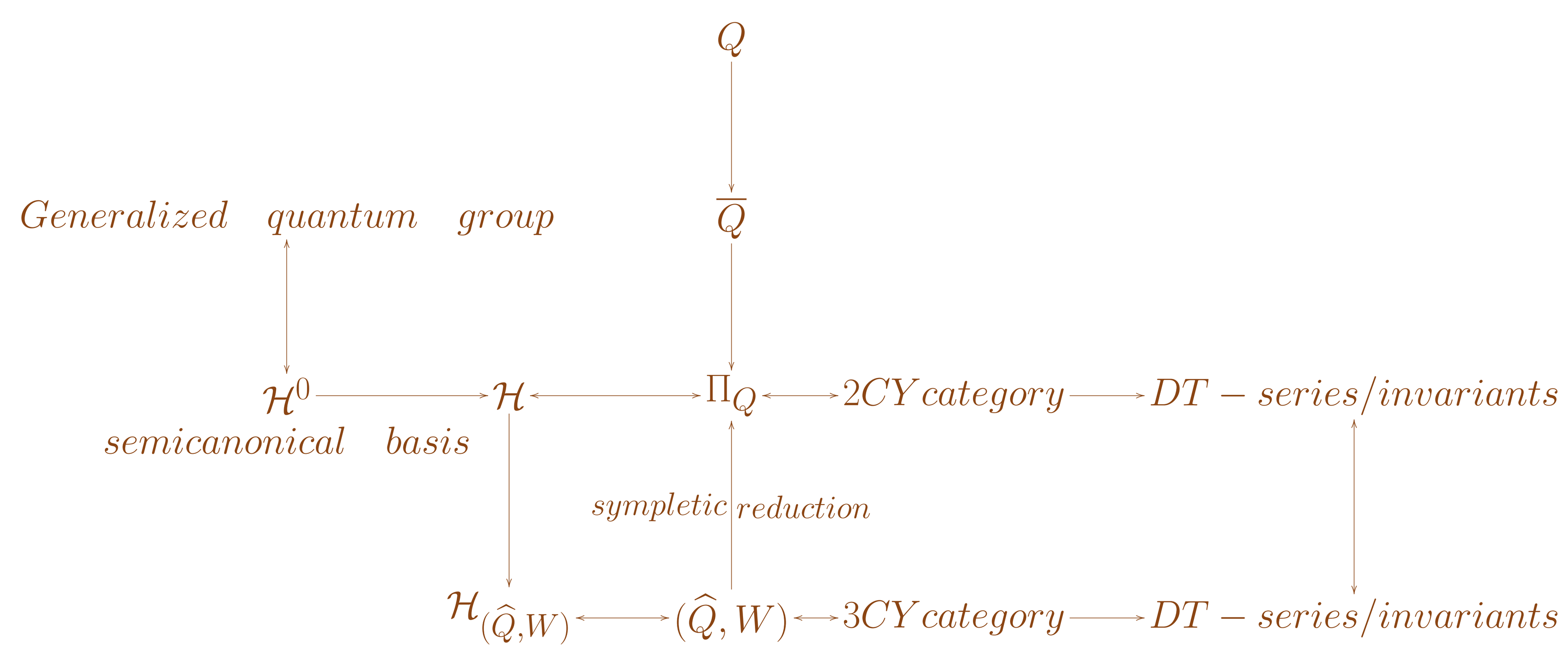
We call $\mathcal{H}$ the cohomological Hall algebra (COHA) of Π_Q. The zero degree part $\mathcal{H}^0 = \bigoplus_{\gamma\in\mathbb{Z}^I_{\geqslant 0}} \mathcal{H}^0_\gamma = \bigoplus_{\gamma\in\mathbb{Z}^I_{\geqslant 0}} H^{-2\chi_Q(\gamma,\gamma)}_{c,\mathbf{G}_\gamma}(\mathbf{M}^{sp}_{\Pi_Q,\gamma},\mathbb{Q})^\vee \otimes \mathbb{T}^{-\chi_Q(\gamma,\gamma)}$ is a subalgebra of $\mathcal{H}$.

The classes of irreducible componets $\{[Z] | Z \in \mathrm{Irr}(\mathbf{M}^{sp}_{\Pi_Q,\gamma})\}$ form a basis of $\mathcal{H}^0$, which is called *semicanonical basis*. It is compatible with a certain filtration.

Abstract

We discuss semicanonical bases from the point of view of Cohomological Hall algebras via the "dimensional reduction" from 3-dimensional Calabi-Yau categories to 2-dimensional ones. Also, we discuss the notion of motivic Donaldson-Thomas invariants (as defined by M. Kontsevich and Y. Soibelman) in the framework of 2-dimensional Calabi-Yau categories. In particular we propose a conjecture which allows one to define Kac polynomials for a 2-dimensional Calabi-Yau category (this is a theorem of S. Mozgovoy in the case of preprojective algebras).

- $Q = (I,\Omega)$: quiver with the set of vertices I and the set of arrows Ω;
- $\overline{Q} = (I, \Omega \sqcup \overline{\Omega})$: the double quiver obtained by adding an inverse arrow $a^* : j \to i$ for any arrow $a : i \to j \in \Omega$;
- $\Pi_Q = \mathbb{C}\overline{Q}/\sum_{a\in\Omega}[a,a^*]$: the preprojective algebra;
- $(\widehat{Q}, W) = \begin{cases}\widehat{Q} = (I, \Omega\sqcup\overline{\Omega}\sqcup\{l_i\})\\ W = \sum_{a\in\Omega}[a,a^*]l,\ l = \sum_{i\in I} l_i.\end{cases}$: the triple quiver $\widehat{Q}$ with potential W, obtained by adding loops $l_i : i \to i$ at each vertex $i \in I$ to $\overline{Q}$.



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2CY categories and Donaldson-Thomas series

- $\mathcal{C}$: ind-constructible locally regular (e.g. locally Artin) triangulated A_∞-category over a field $\mathbf{k}$, whose stack of objects admits a countable decomposition into the union of quotient stacks $\mathcal{O}b(\mathcal{C}) = \sqcup_{i\in I}(Y_i, GL(N_i))$, where Y_i is a reduced algebraic scheme acted by the group $GL(N_i)$;
- $H(\mathcal{C})$: motivic Hall algebra, the $Mot(Spec(\mathbf{k}))$−module $\bigoplus_{i\in I} Mot_{st}(Y_i, GL(N_i))[\mathbb{L}^n, n < 0]$ (Mot_{st} means stack functions);
- $\mathcal{C}_V := \mathcal{C}_{V,Log}$: category generated by semistables with the central charge in a strict sector $V \subset \mathbb{R}^2$, for a fixed class map $cl : K_0(\mathcal{C}) \to \Gamma$, a central charge $Z : \Gamma \to \mathbb{C}$ and a branch Log of the logarithm function on V;
- $\widehat{H}(\mathcal{C}_V) := \prod_{\gamma\in(\Gamma\cap C(V,Z,Q))\cup\{0\}} H(\mathcal{C}_V \cap cl^{-1}(\gamma))$: completed motivic Hall algebra associated with V, which contains an invertible element $A^{Hall}_V = 1 + \cdots = \sum_{i\in I} \mathbf{1}_{(\mathcal{O}b(\mathcal{C}_V)\cap Y_i, GL(N_i))}$;
- $\mathcal{R}_{\Gamma,R} := \bigoplus_{\gamma\in\Gamma} R\cdot\widehat{e}_\gamma$: quantum torus over a commutative unital ring R containing an invertible symbol $\mathbb{L}^{\frac{1}{2}}$, where $\widehat{e}_{\gamma_1}\widehat{e}_{\gamma_2} = \mathbb{L}^{\frac{1}{2}\langle\gamma_1,\gamma_2\rangle}\widehat{e}_{\gamma_1+\gamma_2}$, $\widehat{e}_0 = 1$;
- $\mathcal{R}_{V,R} := \prod_{\gamma\in\Gamma\cap C_0(V,Z,Q)} R\cdot\widehat{e}_\gamma$: quantum torus associated with V, where $C_0(V,Z,Q) := C(V,Z,Q)\cup\{0\}$, and $C(V,Z,Q)$ is the convex cone generated by $S(V,Z,Q) = \{x \in \Gamma_{\mathbb{R}}\setminus\{0\} | Z(x) \in V, Q(x) \geq 0\}$.

The elements A^{Hall}_V satisfy the Factorization Property:

$$A^{Hall}_V = A^{Hall}_{V_1}\cdot A^{Hall}_{V_2}$$

for a strict sector $V = V_1 \sqcup V_2$ (decomposition in the clockwise order).

Results

Define the motivic weight $\omega(E) = \mathbb{L}^{\frac{1}{2}(\chi(E,E))} \in Mot(\mathcal{O}b(\mathcal{C}))$. The map $\Phi : H(\mathcal{C}) \to \mathcal{R}_\Gamma$ given by $\Phi(\nu) = (\nu,\omega)\widehat{e}_\gamma, \nu \in H(\mathcal{C})_\gamma$ satisfies the condition $\Phi(\nu_1\cdot\nu_2) = \Phi(\nu_1)\Phi(\nu_2)$ for $Arg(\gamma_1) > Arg(\gamma_2)$, where $\nu_i \in H(\mathcal{C})_{\gamma_i}$. (here $(\bullet,\bullet)$ is the pairing between motivic measures and motivic functions.) In other words, Φ can be written as $[\pi : Y \to \mathcal{O}b(\mathcal{C})] \mapsto \int_Y \mathbb{L}^{\frac{1}{2}\chi(\pi(y),\pi(y))}\widehat{e}_{cl(\pi(y))}$.

The collections of elements $A^{mot}_V = \Phi(A^{Hall}_V)$ satisfies the Factorization Property: if a strict sector V is decomposed into a disjoint union $V = V_1 \sqcup V_2$ in the clockwise order, then $A^{mot}_V = A^{mot}_{V_1}A^{mot}_{V_2}$.

Motivic DT-series A^{mot}_V is constant on each connected component of the space of stability conditions.

$\Omega(\gamma)$: DT-invariants in $2CY$ case of our DT-series defined by $A^{mot}_V = Sym\left(\sum_{n\geq 0}\mathbb{L}^n\sum_{\gamma\neq 0, Z(\gamma)\in V}\Omega(\gamma)\right.$ $Sym\left(\frac{\sum_{\gamma\neq 0,Z(\gamma)\in V}\Omega(\gamma)\widehat{e}_\gamma}{1-\mathbb{L}}\right)$

Conjecture: There exist elements $a^{mot}_\gamma(\mathbb{L}) \in Mot(Spec(\mathbf{k}))[\mathbb{L}^{\frac{1}{2}}, \mathbb{L}^{-1}, [GL(n)]^{-1}_{n\geqslant 1}]$ which are polynomials in $\mathbb{L}$ and such that the following formula holds in the (commutative) motivic quantum torus:

$$A^{mot}_l = Sym\left(\frac{\sum_{\gamma,Z(\gamma)\in l}(-a^{mot}_\gamma(\mathbb{L})\cdot\mathbb{L})\widehat{e}_\gamma}{1-\mathbb{L}}\right).$$

Furthermore, there exists a $3CY$ category $\mathcal{B}$ such that the elements $a^{mot}(\mathbb{L})$ coincide with motivic DT-invariants with respect to some stability condition on $\mathcal{B}$.