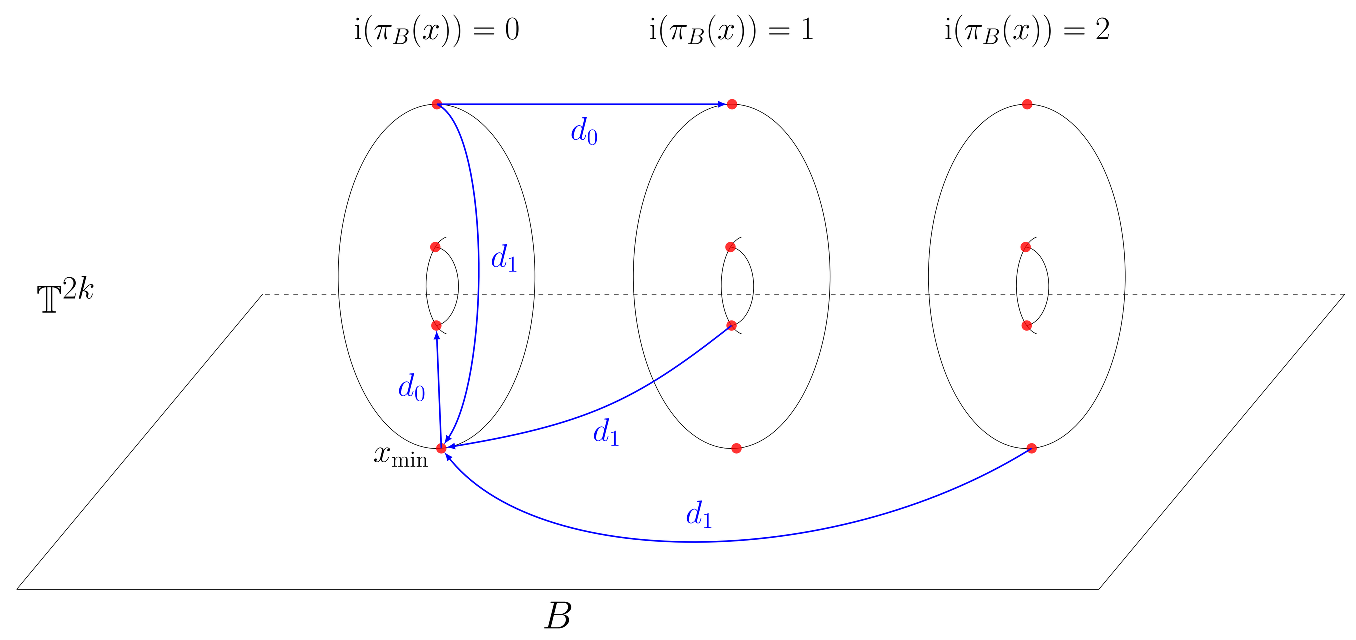
Uniruling of Coisotropic Reductions

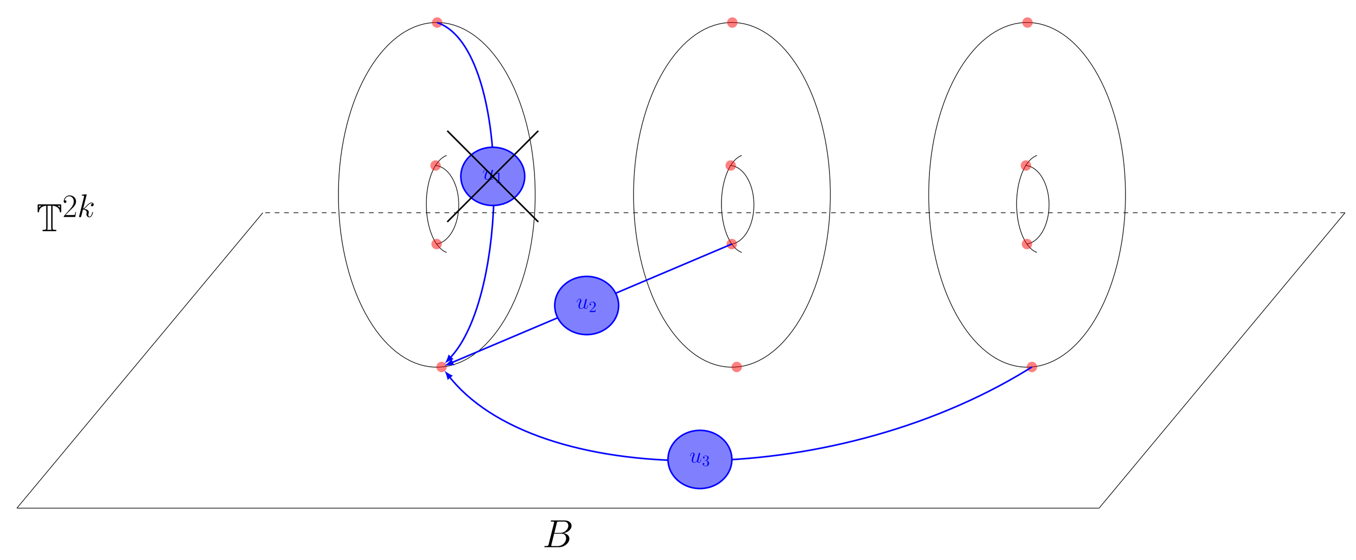
Tobias Sodoge | University College London | tobias.sodoge.13@ucl.ac.uk

Proposition1: Through every point $z_{\min}$ in B there exists a J-holomorphic disc $u:(D,\partial D)\to\big((M\times M,-\omega\times\omega),L_C\big)$ such that the projection of u to B is nonconstant.

Proof: Consider the Morse version of the Leray-Serre Pearl complex for $\mathbb{T}^{2k}\to L_C\to B$.



Since L_C displacable $\Rightarrow$ $HF^*(L_C)=0$. Thus there exist holomorphic discs through the minimum $z_{\min}$ of the Morse function. To ensure that at least on of these projects non-trivially to the base B we have to rule out the possibility that every such disc is contained entirely in the torus fibre $\mathrm{ind}(\pi_B(x)=0)$ over the minimum $z_{\min}$.



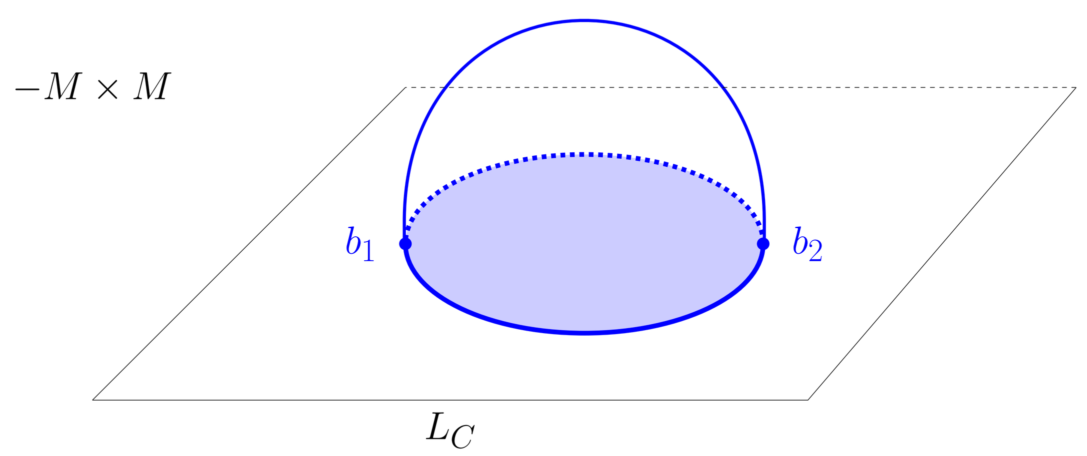
The complex of the $\mathbb{T}^{2k}$ fibre over $z_{\min}$ is generated by the index 1 critical points in this fibre via the Morse intersection product $\star_0$. This product satisfies a Leibniz rule with respect to the pearly Floer differential d:

$$d(x_1\star_0 x_2)=d(x_1)\star_0 x_2+x_1\star_0 d(x_2)$$

The monotonicity and the Minimal Maslov assumption imply that each disc contained entirely in the fibre $\mathrm{ind}(\pi_B(x)=0)$ originates in a critical point y of index at least 2 in this fibre. Now

$$z_{\min}=d(y)=d(x_1\star_0 x_2)=d(x_1)\star_0 x_2+x_1\star_0 d(x_2)$$

yields a contradiction. Thus there have produced a non constant holomorphic disc which as depicted below:



References: See separate list.

Given a regular, stable coisotropic C the graph $L_C=\{(x,y)\in C\times C|\pi_B(x)=\pi_B(y)\}$ is a Lagrangian submanifold of $(M\times M,-\omega\times\omega)$.

Lemma: A regular, stable coisotropic C fits into a torus fibration: $\mathbb{T}^k\to C\to B$ over a symplectic base B.

A coisotropic is regular, if the Leaves L of the characteristic foliation are closed and connected submanifolds of C.

A coisotropic C is stable if there exists a trivialization $Y_1,\dots Y_k$ of the normal bundle of TC such that $\mathcal{L}_{Y_I}i_L^*\omega=0$

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Theorem: Let C be a compact, regular, stable coisotropic submanifold of (M,ω). If the Lagrangian L_C is displaceable, monotone and has minimal Maslov >2, then C is a $\mathbb{T}^k$ fibration over a uniruled base B.

Denote by $X_1,\dots,X_k$ the Reeb vector fields spanning TC^ω. A generalised Reeb orbit is a solution $x(t):S^1\to C$ of the equation:

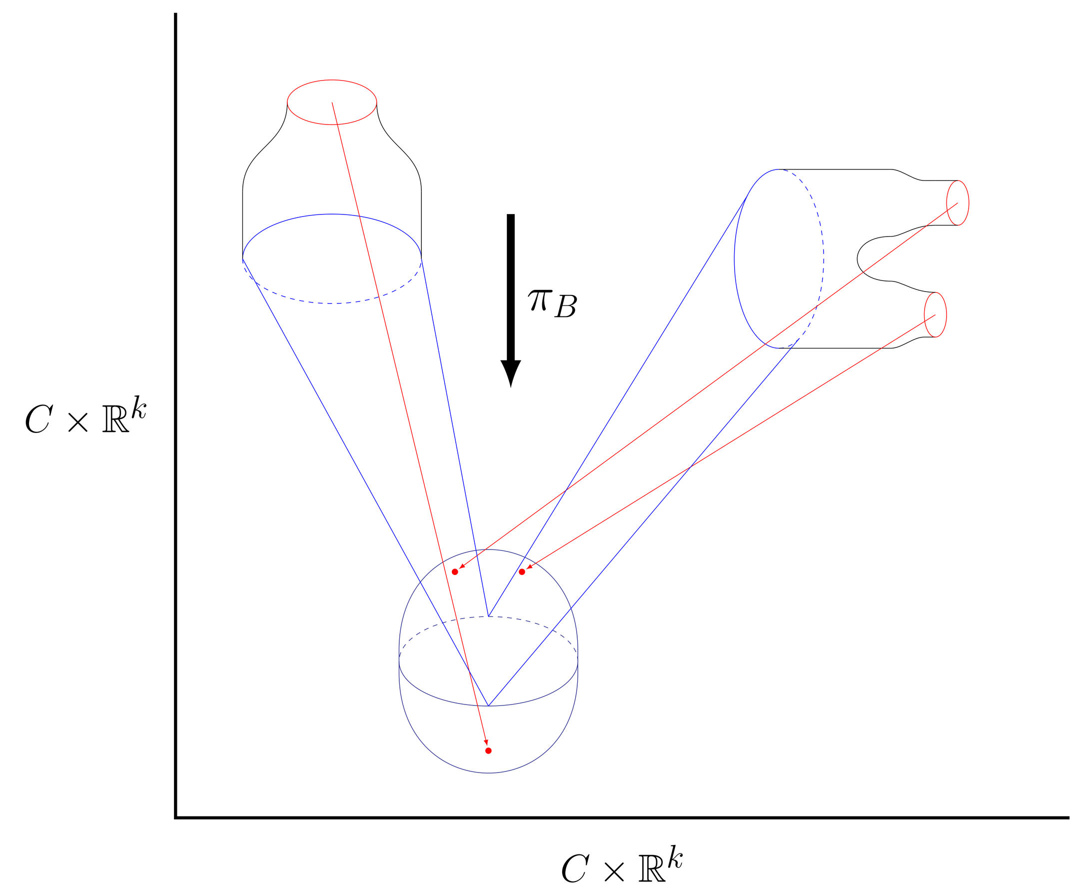
$$\dot{x}(t)=T_1X_1(x(t))+\dots+T_kX_k(x(t))$$

Lemma: If C is stable there exists a neighbourhood U of C symplectomorphic to $(B_\epsilon^k\times C,\omega_s)$, where

$$\omega_s=i_C^*\omega+d\big(p_1\,\iota(Y_1)\omega+\dots+p_k\iota(Y_k)\omega\big)$$

The boundary of U is a stable hypersurface H_C. Reeb orbits on H_C are in 1-1 correspondence with generalised Reeb orbits on C.

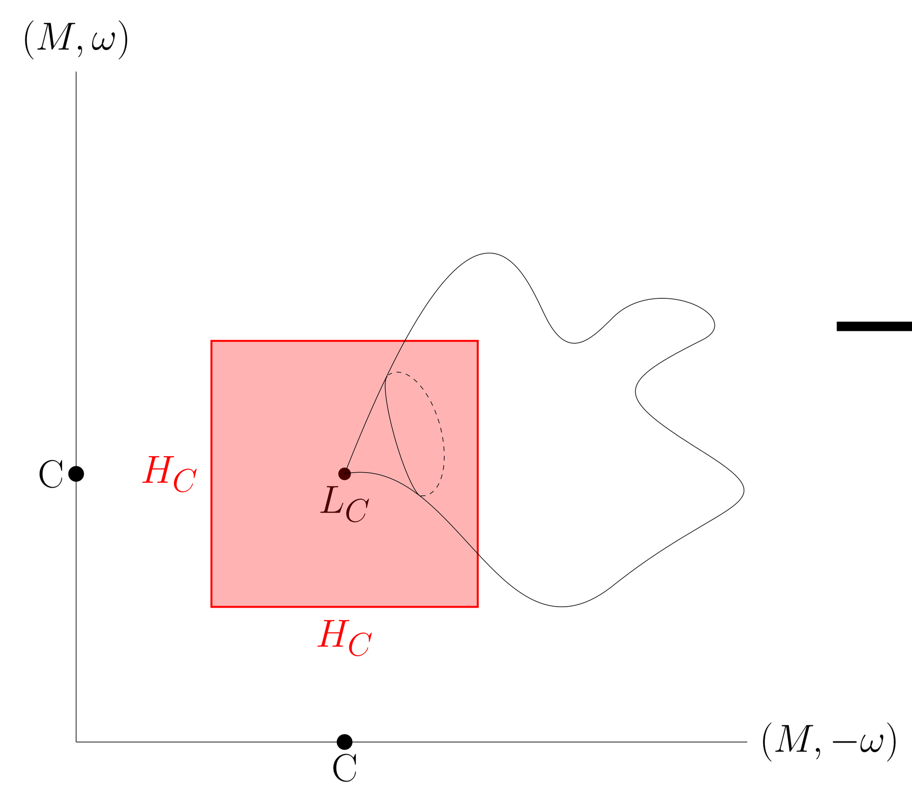
Proposition2: Given neck stretching data adapted to C, we may extract a non-constant holomorphic curves with boundary on L_C and all interior punctures asymptotic to generalised Reeb orbits.



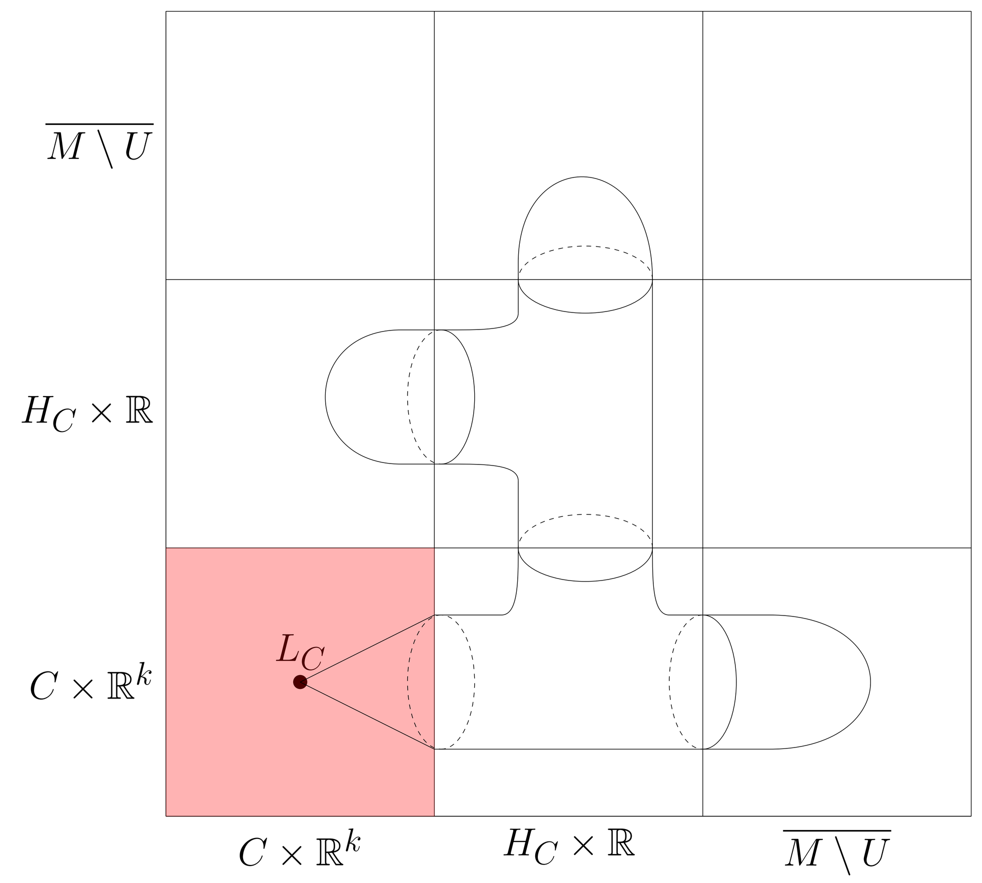
Proof: In the separate factors, the discs have boundaries on C. After projection to B the boundaries match up. The chessboard buildings are asymptotic to generalised Reeb orbits at all punctures. Since B is compact, after projection all punctures become removable.

Proposition3: The projections to the base B of the holomorphic curves are non constant, holomorphic spheres.

A symplectic manifold (B,ω) is uniruled if through every point $b\in B$ there exists a non-constant, J-holomophic sphere $u:S^2\longrightarrow B$.



neck-stretch



Proof: Consider L_C in its neighbourhood U with stable boundary H_C in the product $(M\times M,-\omega\times\omega)$ and the holomorphic disc provided by Proposition 1.

We simultaneously neck stretch around $H_C\times H_C$. Thus we have to be a bit careful in the SFT analysis. We obtain a chessboard holomorphic building with boundary on L_C. Over each level of the holomorphic building in the split parts of $(M,-\omega)$ there is a holomorphic building in the (M,ω) factor and vice versa.