Some results of Hamiltonian homomorphisms on closed aspherical surfaces

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Abstract

On closed symplectically aspherical manifolds, Schwarz proved a classical result that the action function of a nontrivial Hamiltonian homeomorphism is not constant, using Floer homology. In this paper, we generalize Schwarz’s theorem to the \(C^2\)-case on closed aspherical surfaces. Our method is based on a transversal foliation for dynamical systems of surfaces inspired by Le Calvez and its recent progress. As an application, we prove that the contractible fixed points set (and consequently the fixed points set) of a nontrivial Hamiltonian homeomorphism is not connected. Furthermore, we obtain that the growth of the action width of a Hamiltonian homeomorphism increases at least linearly and that the group of Hamiltonian homeomorphisms of \(S^2\) and the group of area preserving homeomorphisms isotopic to the identity of \(\mathbb{C}\) are torsion free.

1. Motivation

The famous Gromov–Eliashberg Theorem, that the group of symplectic diffeomorphisms is \(C^0\)-closed in the full group of diffeomorphisms, makes us interested in defining a symplectic homeomorphism as a homeomorphism which is a \(C^0\)-limit of symplectic diffeomorphisms. This becomes a central theme of what is now called \(C^0\)-symplectic topology. There is a family of problems in symplectic topology that are interesting to be extended to the continuous analogs of classical smooth objects of the symplectic world. In the theory of \(C^0\)-symplectic topology, there are many questions still open, e.g., the \(C^0\)-flow conjecture, and the simplicity of the group of Hamiltonian homeomorphisms of surfaces.

Let \(S\) be a closed oriented surface with genus \(g \geq 1\). In this case, \(S\) is a closed aspherical surface with the property \(\pi_1(S) = 0\). Let \(J = (\mathcal{I}_J)\), be an identity isotopy on \(J\), that is, \(J\) is a continuous path in \(\text{Homeo}(S)\) with \(J_0 = \text{Id}\).

Suppose that its time-one map \(J^t\) preserves the measure \(\mu\) induced by \(J\). It is well known that the condition of the rotation vector of \(\mu\), \(\rho(\mu)\), is \(\rho(\mu) \in H^1(S) \cong \mathbb{R}\), varying is equivalent to saying that the homeomorphism \(J\) is in the homotopy class \(\text{Fix}(\mu)\).

In this sense, we call such a Hamiltonian isotopy and such a \(J\) a Hamiltonian homeomorphism. In this article, we carry out some foundational studies of Hamiltonian homeomorphisms (and a more general notion) on closed aspherical surfaces.

2. Notations

Let \(F\) be the time-one map of an identity isotopy \(I\) on \(S\). We denote by \(\text{Homeo}(S)\) the group of homeomorphisms of \(S\) (resp. \(C^0\)-diffeomorphisms, \(C^\infty\)-diffeomorphisms) of \(S\). Denote by \(\text{Fix}(\mu)\) the set of Borel finite measures on \(S\) that are invariant by \(J\) and have no atoms on \(\text{Homeo}(S)\).

Denote by \(\text{Fix}(\mu)\) the identity component of the topological space of \(\text{Fix}(\mu)\) for the compact-open topology.

We say that a homeomorphism \(f\) is \(\mu\)-symplectic if \(f \in \text{Fix}(\mu)\) has full support. An identity isotopy \(J\) is \(\mu\)-Hamiltonian if the time-one map \(J^t\) is \(\mu\)-symplectic and \(\rho(\mu) = 0\). A homeomorphism \(f\) is \(\mu\)-Hamiltonian if there exists a \(\mu\)-Hamiltonian isotopy \(I\) such that the time-one map of all \(I\) is \(J^t\).

3. Previous Work

Let \((M, \omega)\) be a symplectic manifold with \(\omega|M| = 0\). Suppose that \(M \times \mathbb{R} \cong M \times S^1\) is one-periodic in time, is the Hamiltonian function generating the flow \(I\). Denote by \(\text{Fix}(\omega)(F)\) the set of contractible fixed points of \(F\), that is, \(x \in \text{Fix}(\omega)(F)\) and only if \(x\) is a fixed point of \(F\) and the oriented loop \(I(x) = t \rightarrow F_t(x)\) defined on \([0, \Omega]\) is contractible on \(M\). The classical action function is defined, up to an additive constant, on \(\text{Fix}(\omega)(F)\) as follows:

\[\mathcal{A}(x, \xi) = \int_0^\Omega \omega(F_t^*(\xi), \xi) \ dt,\]

where \(x \in \text{Fix}(\omega)(F)\) and \(\partial_t H|_t = 0\) is any 2-simplex with \(\partial_0 H|_t = 0\). The following deep result was proved [Sc00] by using Floer homology with the real filtration induced by the action function.

Theorem 1 (Schwarz) Let \((M, \omega)\) be a closed symplectic manifold with \(\omega|M| = 0\). Suppose that \(\varphi = \varphi(I)\) is a Hamiltonian flow on \((M, \omega)\) generated by a Hamiltonian function \(H\). Assume that \(\varphi(0) = \varphi(\Omega)\). Then there exist \(a, b \in \text{Fix}(\omega)(F)\) such that \(\mathcal{A}(a) < \mathcal{A}(b)\).

5. Our conclusions

The contributions of this poster can be summarized as follows:

1. In the classical case, one can prove that the action function is a constant on a connected, contractible fixed points by Sard’s theorem. In each of the generalized cases given in Theorem 2, we prove that this property still holds. Our method is merely topological.

2. Given the generalized action function, one may ask whether Schwarz’s theorem is still true. We show in this article that it is true in the second case of Theorem 2 but no longer true when the measure \(\mu\) has no full support even for \(F \in \text{Diff}(S)\). The main tools we use in its proof are the theory of transverse foliations for dynamical systems of surfaces inspired by Le Calvez [Le] and its recent progress [Jau14].

Theorem 4 Let \(J\) be the time-one map of a \(\mu\)-Hamiltonian isotopy \(I\). If \(I\) satisfies the \(\text{WB-property}\) and \(\rho(\mu) = 0\), the action function defined in Theorem 2 is not constant.

3. As an application of Theorem 3 and Theorem 4, we obtain that the contractible fixed points set (and consequently the fixed points set) of a nontrivial Hamiltonian homeomorphism is not connected.

Theorem 5 Let \(J\) be the time-one map of a \(\mu\)-Hamiltonian isotopy \(I\). If \(I\) satisfies the \(\text{WB-property}\) and \(\rho(\mu) = 0\), the action function defined in Theorem 2 is not constant.

We fix a Borel finite measure \(\mu\) which has a full support and has no atoms on \(\mathbb{S}\) (e.g., the measure \(\mu\) induced by the area form \(\Omega\)). Obviously, the set \(\text{Hom}(\mathbb{S}, \mu)\) forms group (the operation is the composition of the maps). Denote by \(\text{Hom}(\mathbb{S}, \mu)\) the subgroup of \(\text{Hom}(\mathbb{S}, \mu)\) whose elements preserve the measure \(\mu\). Denote by \(\text{Hom}(\mathbb{S}, \mu)\) the subset of \(\text{Hom}(\mathbb{S}, \mu)\) which have \(\mu\)-Hamiltonian. It has been proved that \(\text{Hom}(\mathbb{S}, \mu)\) forms a group.

Finally, we obtain that the growth of the action width of a Hamiltonian homeomorphism increases at least linearly, based on which we obtain

Theorem 6 The groups \(\text{Ham}(\mathbb{C}^2, \mu)\) and \(\text{Homeo}(\mathbb{S}, \mu)\) have full support.

References


Symplectic topology, Sheaves and Mirror Symmetry, 2016