Some results of Hamiltonian homeomorphisms on closed aspherical surfaces

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Abstract

On closed symplectically aspherical manifolds, Schwarz proved a classical result that the action function of a nontrivial Hamiltonian diffeomorphism is not constant by using Floer homology. In this poster, we generalize Schwarz's theorem to the C^0-case on closed aspherical surfaces. Our methods involve the theory of transverse foliations for dynamical systems of surfaces inspired by Le Calvez and its recent progresses. As an application, we prove that the contractible fixed points set (and consequently the fixed points set) of a nontrivial Hamiltonian homeomorphism is not connected. Furthermore, we obtain that the growth of the action width of a Hamiltonian homeomorphism increases at least linearly and that the group of Hamiltonian homeomorphisms of $\mathbb{T}^2$ and the group of area preserving homeomorphisms isotopic to the identity of Σ_g ($g>1$) are torsion free.

1. Motivation

The famous Gromov-Eliashberg Theorem, that the group of symplectic diffeomorphisms is C^0-closed in the full group of diffeomorphisms, makes us interested in defining a symplectic homeomorphism as a homeomorphism which is a C^0-limit of symplectic diffeomorphisms. This becomes a central theme of what is now called "C^0-symplectic topology". There is a family of problems in symplectic topology that are interesting to be extended to the continuous analogs of classical smooth objects of the symplectic world. In the theme of C^0-symplectic topology, there are many questions still open, e.g., the C^0-flux conjecture, and the simplicity of the group of Hamiltonian homeomorphisms of surfaces.

Let S be a closed oriented surface with genus $g\geq 1$. In this case, S is a closed aspherical surface with the property $\pi_2(S)=0$. Let $I=(F_t)_{t\in[0,1]}$ be an identity isotopy on S, that is, I is a continuous path in $\mathrm{Homeo}(S)$ with $F_0=\mathrm{Id}_S$. We suppose that its time-one map F preserves the measure μ induced by ω. It is well known that the condition of the rotation vector of μ, $\rho_I(\mu)\in H_1(S,\mathbb{R})\simeq\mathbb{R}^{2g}$, vanishing is equivalent to saying that the homeomorphism F is in the C^0-closure of $\mathrm{Ham}(S,\omega)$. In this sense, we call such I a *Hamiltonian isotopy* and such F a *Hamiltonian homeomorphism*. In this article, we carry out some foundational studies of Hamiltonian homeomorphisms (and a more general notion) on closed aspherical surfaces.

2. Notations

Let F be the time-one map of an identity isotopy I on S. We denote by $\mathrm{Homeo}(S)$ (resp. $\mathrm{Diff}(S)$, $\mathrm{Diff}^1(S)$) the set of homeomorphisms (resp. diffeomorphisms, C^1-diffeomorphisms) of S. Denote by $\mathcal{M}(F)$ the set of Borel finite measures on S that are invariant by F and have no atoms on $\mathrm{Fix}_{\mathrm{Cont},I}(F)$.

Denote by $\mathrm{Homeo}_*(S)$ the identity component of the topological space of $\mathrm{Homeo}(S)$ for the compact-open topology.

We say that a homeomorphism F is *μ-symplectic* if $\mu\in\mathcal{M}(F)$ has full support. An identity isotopy I is *μ-Hamiltonian* if the time-one map F is μ-symplectic and $\rho_I(\mu)=0$. A homeomorphism F is *μ-Hamiltonian* if there exists a μ-Hamiltonian isotopy I such that the time-one map of I is F.

3. Previous Work

Let (M,ω) be a symplectic manifold with $\pi_2(M)=0$. Suppose that $H:\mathbb{R}\times M\to\mathbb{R}$, one-periodic in time, is the Hamiltonian function generating the flow I. Denote by $\mathrm{Fix}_{\mathrm{Cont},I}(F)$ the set of contractible fixed points of F, that is, $x\in\mathrm{Fix}_{\mathrm{Cont},I}(F)$ if and only if x is a fixed point of F and the oriented loop $I(x):t\mapsto F_t(x)$ defined on $[0,1]$ is contractible on M. The classical action function is defined, up to an additive constant, on $\mathrm{Fix}_{\mathrm{Cont},I}(F)$ as follows

$$\mathcal{A}_H(x)=\int_{D_x}\omega-\int_0^1 H(t,F_t(x))\,\mathrm{d}t, \tag{1}$$

where $x\in\mathrm{Fix}_{\mathrm{Cont},I}(F)$ and $D_x\subset M$ is any 2-simplex with $\partial D_x=I(x)$. The following deep result was proved [Scz00] by using Floer homology with the real filtration induced by the action function.

Theorem 1 (Schwarz) *Let (M,ω) be a closed symplectic manifold with $\pi_2(M)=0$. Let $I=(F_t)_{t\in\mathbb{R}}$ be a Hamiltonian flow on M with $F_0=\mathrm{Id}_M$ and $F_1=F$ generated by a Hamiltonian function H. Assume that $F\neq\mathrm{Id}_M$. Then there are $x,y\in\mathrm{Fix}_{\mathrm{Cont},I}(F)$ such that $\mathcal{A}_H(x)\neq\mathcal{A}_H(y)$.*

Through the WB-property (see Definition 1 below), the classical action of Hamiltonian diffeomorphism has been generalized to the case of Hamiltonian homeomorphism (and to more general cases) [Wan11]:

Theorem 2 *Let $F\in\mathrm{Homeo}(S)$ be the time-one map of an identity isotopy I on S. Suppose that $\mu\in\mathcal{M}(F)$ and $\rho_I(\mu)=0$. In each of the following cases:*

- *$F\in\mathrm{Diff}(S)$ (not necessarily C^1);*
- *I satisfies the WB-property and the measure μ has full support;*
- *I satisfies the WB-property and the measure μ is ergodic,*

an action function L_μ can be defined, which generalizes the classical one given by Eq. 1.

4. Definitions

Let M be a surface homeomorphic to the complex plane $\mathbb{C}$ and let $I=(F_t)_{t\in[0,1]}$ be an identity isotopy on M. For every two different fixed points z and z' of F_1, the *linking number* $i_I(z,z')\in\mathbb{Z}$ is the degree of the map $\xi:\mathbf{S}^1\to\mathbf{S}^1$ defined by

$$\xi(e^{2i\pi t})=\frac{h\circ F_t(z')-h\circ F_t(z)}{|h\circ F_t(z')-h\circ F_t(z)|},$$

where $h:M\to\mathbb{C}$ is a homeomorphism. The linking number is independent of h.

Let F be the time-one map of an identity isotopy $I=(F_t)_{t\in[0,1]}$ on a closed oriented surface S of genus $g\geq 1$, and $\widetilde{F}$ be the time-one map of the lifted identity isotopy $\widetilde{I}=(\widetilde{F}_t)_{t\in[0,1]}$ of I on the universal cover $\widetilde{S}$ of S. Denote by $\widetilde{\Delta}$ the diagonal of $\mathrm{Fix}(\widetilde{F})\times\mathrm{Fix}(\widetilde{F})$.

When $g>1$, it is well known that the fundamental group $\pi_1(\mathrm{Homeo}_*(S))$ is trivial. It implies that any two identity isotopies $I,I'\subset\mathrm{Homeo}_*(S)$ with fixed endpoints are homotopic. Hence, I is unique up to homotopy, which implies that $\widetilde{F}$ is uniquely defined and independent of the choice of the isotopy from Id_S to F. When $g=1$, $\widetilde{F}$ depends on the isotopy I since $\pi_1(\mathrm{Homeo}_*(S))\simeq\mathbb{Z}^2$.

Note that the universal cover $\widetilde{S}$ is homeomorphic to $\mathbb{C}$. We define the *linking number* $i(\widetilde{F};\widetilde{z},\widetilde{z}')$ for each pair $(\widetilde{z},\widetilde{z}')\in(\mathrm{Fix}(\widetilde{F})\times\mathrm{Fix}(\widetilde{F}))\setminus\widetilde{\Delta}$ as

$$i(\widetilde{F};\widetilde{z},\widetilde{z}')=i_{\widetilde{I}}(\widetilde{z},\widetilde{z}'). \tag{2}$$

Definition 1 (WB-property and B-property) *We say that I satisfies* the weak boundedness property at $\widetilde{a}\in\mathrm{Fix}(\widetilde{F})$ (WB-property at $\widetilde{a}$) *if $i(\widetilde{F};\widetilde{a},\widetilde{b})$ is uniformly bounded for all fixed point $\widetilde{b}\in\mathrm{Fix}(\widetilde{F})\setminus\{\widetilde{a}\}$. We say that I satisfies* the weak boundedness property (WB-property) *if it satisfies the weak boundedness property at every $\widetilde{a}\in\mathrm{Fix}(\widetilde{F})$. We say that I satisfies* the boundedness property (B-property) *if the set of $i(\widetilde{F};\widetilde{a},\widetilde{b})$ where $(\widetilde{a},\widetilde{b})\in(\mathrm{Fix}(\widetilde{F})\times\mathrm{Fix}(\widetilde{F}))\setminus\widetilde{\Delta}$ is bounded.*

Obviously, the B-property implies the WB-property. It has been proved that the WB-property is satisfied if $F\in\mathrm{Diff}(S)$ and that the B-property is satisfied if $F\in\mathrm{Diff}^1(S)$ [Wan11]. Moreover, the set of all WB-property points of I is shown dense in $\mathrm{Fix}(\widetilde{F})$ [Ler14]. We prove that I satisfies the B-property if the number of the connected components of $\mathrm{Fix}_{\mathrm{Cont},I}(F)$ is finite.

Definition 2 (Action spectrum and Action width) *Under the same hypotheses as Theorem 2, we define the* action spectrum of I *(up to an additive constant):*

$$\sigma(I)=\{L_\mu(z)\mid z\in\mathrm{Fix}_{\mathrm{Cont},I}(F)\}\subset\mathbb{R},$$

and the following action width of I*:*

$$\mathrm{width}(I)=\sup_{x,y\in\sigma(I)}|x-y|.$$

It turns out that the action spectrum $\sigma(I)$ (and hence $\mathrm{width}(I)$) is invariant by conjugation in $\mathrm{Homeo}^+(M,\mu)$, where $\mathrm{Homeo}^+(M,\mu)$ is the subgroup of $\mathrm{Homeo}(M)$ whose elements preserve the measure μ and the orientation. Moreover, the action function L_μ only depends on the homotopic class with fixed endpoints of I, so do $\sigma(I)$ and $\mathrm{width}(I)$. Hence we can simply write $\sigma(F)$ (resp. $\mathrm{width}(F)$) instead of $\sigma(I)$ (resp. $\mathrm{width}(I)$).

5. Our conclusions

The contributions of this poster can be summarized as following:

1. In the classical case, one can prove that the action function is a constant on a connected set of contractible fixed points by Sard's theorem. In each of the generalized cases given in Theorem 2, we prove that this property still holds. Our method is purely topological.

Theorem 3 *Under the hypotheses of Theorem 2, the action function defined in Theorem 2 is a constant on each connected component of $\mathrm{Fix}_{\mathrm{Cont},I}(F)$.*

2. Given the generalized action function, one may ask whether Schwarz's theorem is still true. We show in this article that it is true in the second case of Theorem 2 but no longer true when the measure μ has no full support even for $F\in\mathrm{Diff}(S)$. The main tools we use in its proof are the theory of transverse foliations for dynamical systems of surfaces inspired by Le Calvez [Lec05] and its recent progress [Jau14].

Theorem 4 *Let F be the time-one map of a μ-Hamiltonian isotopy I. If I satisfies the WB-property and $F\neq\mathrm{Id}$, the action function defined in Theorem 2 is not constant.*

3. As an application of Theorem 3 and Theorem 4, we obtain that the contractible fixed points set (and consequently the fixed points set) of a nontrivial Hamiltonian homeomorphism is not connected.

Theorem 5 *Let F be the time-one map of a μ-Hamiltonian isotopy I. If the set $\mathrm{Fix}_{\mathrm{Cont},I}(F)$ is connected, then F must be Id_M. In particular, if $\mathrm{Fix}(F)$ is connected, then F must be Id_M.*

Remark 1 *If $F\neq\mathrm{Id}_M$, Theorem 5 implies that the number of connected components of the set $\mathrm{Fix}_{\mathrm{Cont},I}(F)$ is at least two, which is optimal. If $\mathrm{Fix}_{\mathrm{Cont},I}(F)$ has exactly two connected components, its cardinality must be infinite.*

Remark 2 *This property is merely a 2-dimension phenomenon. This is because Buhovsky et al. [BHS16] have recently constructed a Hamiltonian homeomorphism with only one fixed point on any closed symplectic manifold of dimension at least four.*

4. We fix a Borel finite measure μ which has a full support and has no atoms on S (e.g., the measure μ induced by the area form ω). Obviously, the sets $\mathrm{Homeo}(S)$ and $\mathrm{Homeo}_*(S)$ form groups (the operation is the composition of the maps). Denote by $\mathrm{Homeo}_*(S,\mu)$ the subgroup of $\mathrm{Homeo}_*(S)$ whose elements preserve the measure μ. Denote by $\mathrm{Hameo}(S,\mu)$ the subset of $\mathrm{Homeo}_*(S,\mu)$ whose elements are μ-Hamiltonian. It has been proved that $\mathrm{Hameo}(S,\mu)$ forms a group.

Finally, we obtain that the growth of the action width of a Hamiltonian homeomorphism increases at least linearly, based on what we obtain

Theorem 6 *The groups $\mathrm{Hameo}(\mathbb{T}^2,\mu)$ and $\mathrm{Homeo}_*(\Sigma_g,\mu)$ ($g>1$) are torsion free, where μ has full support.*

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