Motivic Hilbert Zeta Functions of Curves

Dori Bejleri
Brown University

Zeta functions in algebraic geometry

Let X be a variety over $\mathbb{F}_q$. Denote $N_m = \#X(\mathbb{F}_{q^m})$. The ***Hasse-Weil zeta function*** of X is defined as the formal power series

$$Z_X(t) := \exp\left(\sum_{m\geq 1} \frac{N_m}{m} t^m\right)$$

Dwork proved the following remarkable theorem:

$$Z_X(t) \in \mathbb{Q}(t);$$

that is, the Hasse-Weil zeta function is a rational function. Rationality of the Hasse-Weil zeta function is the first in a series of conjectures posed by Weil and first proved for smooth curves. The first two are as follows:

The Weil Conjectures for curves

Let C be a smooth projective curve of genus g over $\mathbb{F}_q$. Then:

- There is a polynomial $P(t)$ of degree $2g$ such that

$$Z_C(t) = \frac{P(t)}{(1-t)(1-qt)}$$

- $Z_C(t)$ satisfies the functional equation

$$Z_C(t) = (qt^2)^{g-1} Z_C(1/qt)$$

Key Observation: Letting $\mathrm{Sym}^n(X) = X^n/\mathfrak{S}_n$ be the space of effective zero cycles, then

$$Z_X(t) = \sum_{n\geq 0} \#\,\mathrm{Sym}^n(X)(\mathbb{F}_q) t^n \in \mathbb{Z}[\![t]\!]$$

This motivates the definition of the ***Kapranov motivic zeta function***

$$Z_X^{\mathrm{Sym}}(t) := \sum_{n\geq 0} [\mathrm{Sym}^n(X)] t^n \in 1 + tK_0(\mathrm{Var}_k)[\![t]\!]$$

for any variety X over a field k. Here $[\mathrm{Sym}^n(X)]$ denotes the class in the Grothndieck ring of varieties $K_0(\mathrm{Var}_k)$:

Grothendieck ring of varieties

$K_0(\mathrm{Var}_k)$ is the ring generated by isomorphism classes $[X]$ of varieties X/k. There are *cut-and-paste relations* given by

$$[X] = [U] + [X \setminus U]$$

whenever $U \subset X$ is open. The product is given by

$$[X][Y] = [X \times Y]$$

We denote the class of $\mathbb{A}^1$ by $\mathbb{L}$.

Let C be a smooth projective irreducible curve of genus g. Assuming $C(k) \neq \emptyset$, Kapranov [Kap00] proved that $(1-t)(1-\mathbb{L}t)Z_C^{\mathrm{Sym}}(t)$ is a polynomial in $K_0(\mathrm{Var}_k)[t]$ of degree $2g$ and there is a functional equation

$$Z_C^{\mathrm{Sym}}(t) = (\mathbb{L}t^2)^{g-1} Z_C^{\mathrm{Sym}}(1/\mathbb{L}t).$$

Litt [Lit15] showed that rationality of $Z_C^{\mathrm{Sym}}(t)$ still holds without the existence of a k-rational point.

Motivic Hilbert zeta functions

The motivic zeta function doesn't see the structure of singularities. For example:

- $Z_X^{\mathrm{Sym}}(t) = Z_{X^{red}}^{\mathrm{Sym}}(t)$
- $Z_C^{\mathrm{Sym}}(t) = Z_{\widetilde{C}}^{\mathrm{Sym}}(t)$ for C a curve with unibranch singularities and $\nu : \widetilde{C} \to C$ is the normalization.

To get an invariant of the singularities, we consider Hilbert schemes instead of symmetric powers. Let X be a scheme and $Y \subset X$ a closed subscheme. Consider

$$\mathrm{Hilb}^n_Y(X) := \left\{ \begin{matrix} \text{zero-dimensional subschemes } Z \subset X \\ \dim_k \mathcal{O}_Z = d \;\;\&\;\; \mathrm{Supp}(Z) \subset Y \end{matrix} \right\}$$

When $Y = X$, $\mathrm{Hilb}^n_Y(X) = \mathrm{Hilb}^n(X)$ is the usual Hilbert scheme of points. We define the ***motivic Hilbert zeta function*** of $Y \subset X$ as

$$Z_{Y/X}^{\mathrm{Hilb}}(t) := \sum_{n\geq 0} [\mathrm{Hilb}^n_Y(X)] t^n \in 1 + tK_0(\mathrm{Var}_k)[\![t]\!]$$

and denote $Z_{X/X}^{\mathrm{Hilb}}(t) =: Z_X^{\mathrm{Hilb}}(t)$.

Key properties:

- If $U \subset X$ is an open subscheme complement Z then

$$Z_X^{\mathrm{Hilb}}(t) = Z_U^{\mathrm{Hilb}}(t) Z_{Z/X}^{\mathrm{Hilb}}(t)$$

- If C is a smooth curve then

$$Z_C^{\mathrm{Hilb}}(t) = Z_C^{\mathrm{Sym}}(t)$$

Main result

Theorem(B.-Ranganathan-Vakil)

Let C be a *generically planar curve* with k-rational non-planar singularities. Then $Z_C^{\mathrm{Hilb}}(t)$ is a rational function. In particular, this holds if C is a *reduced curve* with k-rational singularities.

Idea of proof: Suppose C is reduced. Using cut-and-paste reduce to $Z_{p/C}^{\mathrm{Hilb}}(t)$ where $p \in C$ is one of (finitely many) singularities. The normalization $\varphi : \widetilde{C} \to C$ is a disjoint union of smooth branches B_i and let $\varphi_i : B_i \to C$ the restriction. We stratify $\mathrm{Hilb}^n_p(C)$ based how far along each branch a subscheme grows:

$$\mathrm{Hilb}_p^{n, a_1, \ldots, a_s}(C) := \{ Z \in \mathrm{Hilb}^n_p(C) \;\;:\;\; \dim_k \varphi_i^* \mathcal{O}_Z = a_i \}$$

The key steps are **i)** the difference $n - \sum_i a_i$ for which $\mathrm{Hilb}_p^{n,a_1,\ldots,a_s}(C) \neq \emptyset$ is bounded **ii)** the class in $K_0(\mathrm{Var}_k)$ of the strata $\mathrm{Hilb}_p^{n,a_1,\ldots,a_s}(C)$ stabalize for a_i large. When some branches are nonreduced planar ribbons, we can proceed similar to below.

Example: monomial plane curves

If $(C, O) \subset \mathbb{A}^2$ is a planar curve singularity defined by a monomial ideal, then $\mathrm{Hilb}^n_O(C)$ inherits an action by $(\mathbb{C}^*)^2$. In this case we can use torus localization techniques to compute $Z_{O/C}^{\mathrm{Hilb}}(t)$. For example:

$$Z_{O/C_n}^{\mathrm{Hilb}}(t) = 1 + \sum_{s=1}^{n} \mathbb{L}^{s-1} t^s \prod_{m=1}^{s} \left(\frac{1}{1 - \mathbb{L}^{s-1} t^s} \right) + \frac{\mathbb{L}^n t^{n+1}}{1-t} \prod_{m=1}^{n} \left(\frac{1}{1 - \mathbb{L}^{m-1} t^m} \right)$$

where $C_n = \{ xy^n = 0 \}$.

Further Questions

Can one describe the coefficients of the numerator and denominator of $Z_{p/C}^{\mathrm{Hilb}}$?

- The denominator should encode the number and nonreduced multiplicity of the branches at p.
- The numerator should be related to (sub)motives of various other moduli spaces of sheaves attached to C.

Planar curves:

The case of reduced planar singularities has been extensively studied in the literature. Rationality and the functional equation follow from studying the map

$$\mathrm{AJ}^n : \mathrm{Hilb}^n(C) \to \overline{\mathrm{Jac}}(C)$$

to the compactified Jacobian and applying Riemann-Roch and Serre duality. After passing to Hodge structures, the numerator of $Z_C^{\mathrm{Hilb}}(t)$ can be described in terms of a perverse filtration on $H^*(\overline{\mathrm{Jac}}(C))$ [MS13] [MY14] and a refinement of $Z_C^{\mathrm{Hilb}}(t)$ recovers the HOMFLY polynomial of algebraic links after specializing to the Euler characteristic [Mau16] [OS12].

- Is $Z_{p/C}^{\mathrm{Hilb}}(t)$ a rational function in $\mathbb{L}$ for (C, p) planar?
- Is there a *blowup formula* relating $Z_{p/C}^{\mathrm{Hilb}}(t)$ to an embedded resolution?
- Is $Z_{p/C}^{\mathrm{Hilb}}(t)$ a constructible function on versal deformations of (C, p)?

Threefold curves:

Much of the literature on Hilbert scheme invariants of a planar curve C involves realizing C as a spectral curve and using various dualities to connect to curve counting theories in 3-folds. For 3-fold singularities we can consider the following. Let Σ be a smooth curve with line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ and define

$$\mathcal{M}_{\mathcal{L}_1, \mathcal{L}_2} := \{ (\mathcal{E}, \varphi_1, \varphi_2) \;\; : \mathcal{E} \text{ a vector bundle }, \varphi_i \in \mathrm{End}(\mathcal{E}) \otimes \mathcal{L}_i \text{ commuting} \}.$$

This is a commuting version of the Hitchin moduli space whose spectral curves have 3-fold singularities. It is natural to ask if there are relations between the Hilbert zeta function of the spectral curves, invariants of $\mathcal{M}_{\mathcal{L}_1, \mathcal{L}_2}$, local curve counting theories on $\mathrm{Tot}_\Sigma(\mathcal{L}_2 \oplus \mathcal{L}_2)$, (quasi)maps to $\mathrm{Hilb}^n(\mathbb{A}^2)$, and affine commuting varieties.

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