Locally arboreal spaces and symplectic structures

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Introduction

The objective of this research project is to construct (shifted) symplectic structures on several moduli spaces parametrizing topological data on manifolds with various decorations. This is a vast generalization of the construction of symplectic structures on moduli spaces of local systems on surfaces.

Theorem 1. *Let $(\mathbb{X}, \mathcal{O})$ be a locally arboreal space with boundary $\partial\mathbb{X}$. An orientation of $(\mathbb{X}, \mathcal{O})$ induces a Lagrangian structure of degree $3 - \dim \mathbb{X}$ on the morphism of moduli spaces of objects $M(\mathcal{O}(\mathbb{X})) \to M(\mathcal{O}(\partial\mathbb{X}))$.*

This is joint work with Vivek Shende.

Locally arboreal spaces

The combinatorial models for Legendrian singularities developed by Nadler associate a stratified topological space $\mathbb{T}$ to a tree T. The arboreal singularity $\mathbb{T}$ has top dimension $|T|$.

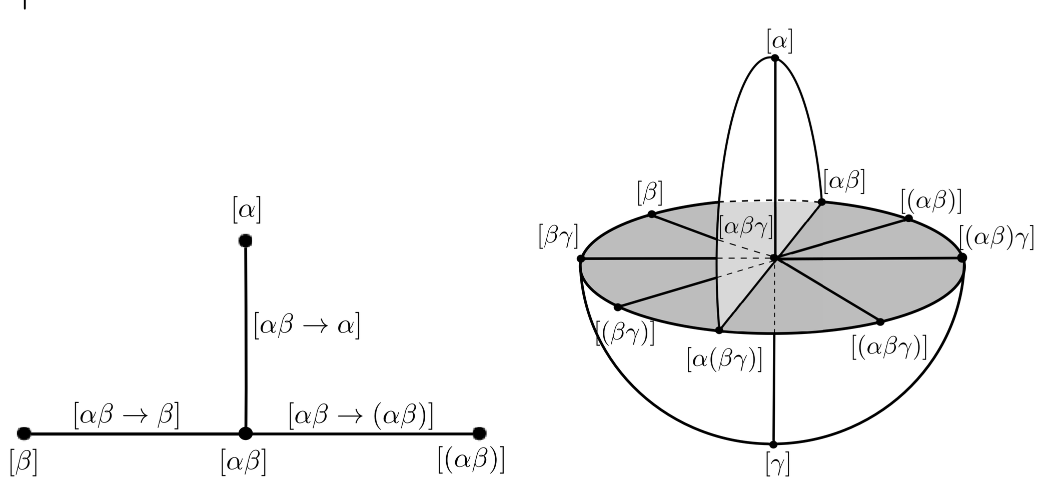

Fig 1: Arboreal singularities for the quivers A_2 and A_3, with labels $\alpha \to \beta$ and $\alpha \to \beta \leftarrow \gamma$. The strata are labelled by correspondences of trees ($R \overset{q}{\leftarrow} P \overset{i}{\hookrightarrow} T$), where parentheses mean that the vertices get identified in the quotient q.

A locally arboreal space $\mathbb{X}$ is locally homeomorphic to $\mathbb{T} \times \mathbb{R}^n$ for some choices of T and n. These arboreal singularities must be glued appropriately; this can be encoded in a sheaf of categories $\mathcal{O}$ on $\mathbb{X}$, locally modeled on the sheaves of categories $\mathcal{N}_T$.

Shifted symplectic structures and categorical orientations

The moduli spaces we consider are naturally derived stacks, with certain finiteness conditions. Such a stack X comes with tangent and cotangent complexes. A n-shifted symplectic form is a closed form of degree n on X, i.e. an element of $\Omega^{2,cl}(X, n)$, which induces an isomorphism $\mathbb{T}_X \to \mathbb{L}_X[-n]$. An degree d orientation on a dg category $\mathcal{C}$ is a map

$$HH_*(\mathcal{C}) \to \mathbb{C}[-d]$$

satisfying some nondegeneracy conditions, where $HH_*(\mathcal{C})$ denotes Hochschild homology. A nondegenerate degree d orientation on a dg category $\mathcal{C}$ induces a $(2-d)$-shifted symplectic structure on the moduli of objects $\mathcal{M}_{\mathcal{C}}$.

Local orientations

Consider the dualizing complex $\omega_{\mathbb{X}}$ of the stratified space $\mathbb{X}$. This is a bounded complex in general, and in particular for a smooth d-manifold it is a shift of the orientation line bundle $\omega_{\mathbb{X}} = \underline{or}[d]$.

A locally arboreal space also comes equipped with a sheaf of categories $\mathcal{O}$. Consider the sheaf of Hochschild homologies $\mathcal{HH}(\mathcal{O})$ which is the sheafification of the presheaf given by $U \mapsto HH_*(\mathcal{O}(U))$.

Definition 2. A *local orientation* of degree d on a locally arboreal space $(\mathbb{X}, \mathcal{O})$ is an isomorphism

$$\mathcal{HH}(\mathcal{O}) \to \omega_{\mathbb{X}}[-d]$$

In the smooth manifold case this is an isomorphism $\underline{\mathbb{C}} \cong \underline{or}$, so an orientation in the classical sense. There is also a notion of nondegeneracy localizing the notion for categorical orientations.

Proposition 3. *Assume $(\mathbb{X}, \mathcal{O})$ be a locally arboreal space with boundary ∂X, and let $\mathcal{HH}_*(\mathcal{C}) \to \omega_{\mathbb{X}}[-d]$ be a nondegenerate local orientation. Then the induced relative orientation on the $HH_*(\partial) \to \mathbb{C}[-d]$ is nondegenerate and gives a Lagrangian structure on the map of moduli spaces $M(\mathcal{O}(\mathbb{X})) \to M(\mathcal{O}(\partial\mathbb{X}))$.*

Examples

Immersed front projections

The main applications will be example coming from microlocal sheaves on manifolds. Consider a d-manifold M and a smooth Legendrian $\Lambda \subset T^\infty M$ at infinity. This defines a singular Lagrangian $\mathbb{X} = M \cup \mathbb{R}_+\Lambda$ inside of T^*M. Such a Lagrangian comes with the Kashiwara-Schapira sheaf of categories μloc, whose local sections are derived categories of microlocal sheaves. We will say that Λ has normal crossings front projection when the singularities of $\mathbb{X}$ are locally diffeomorphic to a union of some coordinate hyperplanes. Such a space is a locally arboreal space.

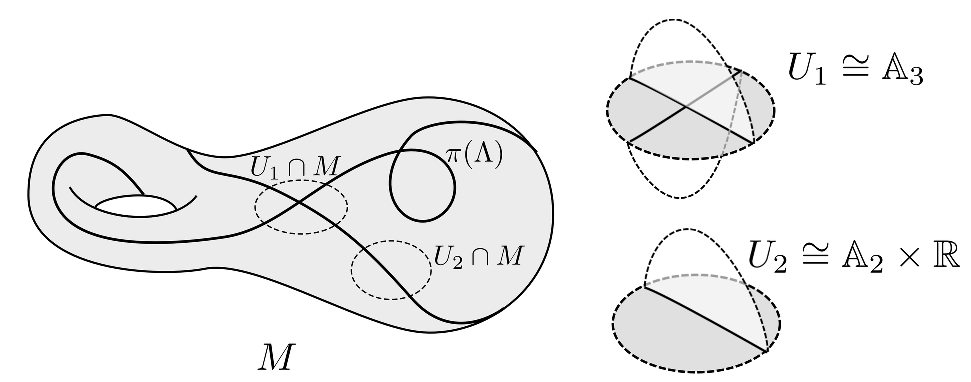

Fig 2: The locally arboreal space corresponding to a smooth Legendrian with normal crossings front projection. We have neighborhoods homeomorphic to $\mathbb{A}_1 \times \mathbb{R}^2$ (smooth locus), $\mathbb{A}_2 \times \mathbb{R}$ and $\mathbb{A}_3$

Theorem 4. *Consider M an* oriented *d-manifold and Λ a smooth Legendrian with normal crossings front projection. Then the locally arboreal space $(\mathbb{X}, \mu loc)$ has a nondegenerate local orientation $\mathcal{HH}(\mu loc) \to \omega_{\mathbb{X}}[-d]$ extending the orientation on M.*

Associated graded of a filtration

Choosing $\mathbb{X}$ to be a comb, i.e. the union of $\mathbb{R}$ and the positive conormal to n points $p_1, \ldots, p_n$.

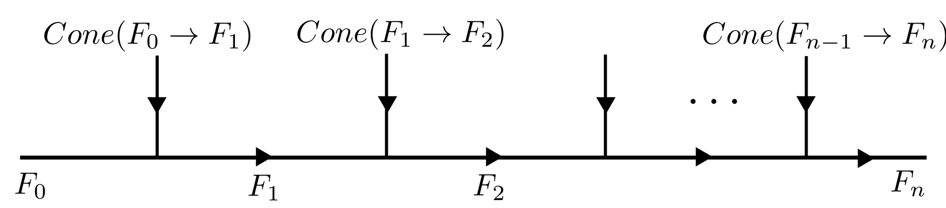

Fig 3: The comb $\mathbb{X}$. The global category $Filt_n$ has objects given by sequences of complexes $F_0 \to \cdots \to F_n$

The category $Sh_{\{p_i\}}(\mathbb{R})$ is the category of n-step filtered complexes $Filt_n$, and the restriction to the boundary

$$Filt_n \to Perf^{n+1} \times Perf$$

is given by taking the associated graded complexes. This map has a degree 2 relative orientation, which induces a Lagrangian structure at the level of moduli spaces. Taking the truncation of the stacks and fixing rank conditions we can interpret this as a Lagrangian correspondence

$$BG \swarrow BP \searrow BL$$

for $G = GL_m$, $L \subset P$ a Levi and a parabolic, where the classifying stacks BG, BP, BL carry canonical 2-shifted symplectic structures. This correspondence has been used in the context of generalizations of the "symplectic implosion" construction.

Wild character varieties

The symplectic structure on wild character varieties on surfaces can also be constructed in this way. Here the relevant space is a surface Σ with some points $p_1, \ldots, p_r$ and around each p_i a Legendrian link $\Lambda \subseteq T^\infty\Sigma$, such that the front projection is the closure of a positive braid.

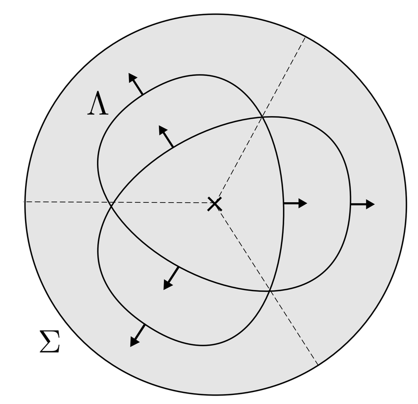

Fig 3: The projection of the Legendrian knot appearing in the description of the wild character variety. The smooth Legendrian circle is obtained by giving a coorientation of the knot projection

The moduli space of microlocal sheaves on such a space can be identified with a "Betti moduli space" version of the moduli of irregular connections on Σ with prescribed singularity types, the *wild character variety*. So we get a Lagrangian morphism of moduli spaces

$$\mathcal{M}_\Lambda \to Loc(S^1)^N$$

given by restriction to the boundary, and in this context it is the map from the wild character variety to local systems on the components of Λ given by taking the "formal monodromies" of the irregular connection.

References

V. Shende and A. Takeda, *Symplectic structures from topological Fukaya categories*, arXiv:1605.02721

D. Nadler, *Arboreal Singularities*, arxiv:1309.4122

V. Shende, D. Treumann, H. Williams, and E. Zaslow, *Cluster Varieties from Legendrian Knots*, arxiv:1512.08942

B. Toën, *Derived algebraic geometry*, EMS Surv. Math. Sci. **1** (2014), 153–240